Efficient Formulations for Dynamic Warehouse Location under Discrete Transportation Costs

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Abstract

A Mixed-integer Linear Programming model is proposed to determine the optimal number, location and capacity of the warehouses required to support a long-term forecast for a business with seasonal demand. Discrete transportation costs, dynamic warehouse contracting, and the handling of safety stock are the three main distinctive features of the problem. Four alternatives for addressing discrete transportation costs are compared. The most efficient formulation is obtained using integer variables to account for the number of units used of each transportation mode. Contracting policies constraints are derived to ensure use of warehouses for continuous periods. Similar constraints are included for the case when a warehouse is closed. Safety stock with risk-pooling effect is considered using a piecewise-linear representation. To solve large-scale problems, tightening constraints, and simplified formulations are proposed. These formulations are based on single-sourcing assumptions and yield near-optimal results with a large reduction in the solution time and a low impact on the optimum cost.

1 Introduction

Supply chains have become increasingly complex in recent years. Globalization has made a large number of new markets and sourcing options available. The response from many companies to this situation has been to focus on their core business, outsourcing the logistics and warehousing operations. Strategic decisions can have a large impact in the success of a company. This is why such decisions must be made using the best tools available.

A Mixed-integer Linear Programming (MILP) model that includes discrete transportation costs, dynamic warehousing contracting policies and safety stock with risk-pooling effect (Eppen, 1979) is proposed in this paper. These features are especially important when the logistics and warehousing operations are outsourced. The goal of the proposed model is to determine the optimal number, location and size of warehouses in a supply chain for a business with seasonal demand. This is not a trivial task because decisions on production of each commodity, transportation mode selection, flows between facilities and inventory must be simultaneously optimized. The features considered make the model more realistic but at the same time significantly harder to solve. This is why efficient model formulations and solution strategies must be developed. This paper focuses on developing efficient formulations for the problem and its solution.

Transportation costs are commonly discrete in practice. The rates are usually fixed over prespecified weight intervals. This important fact has been neglected in most of the previous work presented on the facility location problem (Cornuéjols et al., 1983). Only a handful of problems consider discrete transportation costs using different approaches (Bravo and Vidal, 2013). In this paper, four alternatives to model discrete transportation costs are presented and compared to yield the most efficient formulation for the current application. Section 4.2 details this analysis.

Another important feature considered is warehouse contracting policies. Since the inventory storage service is supplied by an external company, constraints that ensure a continuous service must be enforced. The constraints are derived from propositional logic and impose two conditions: 1) Once a warehouse is opened it must remain open by at least a certain length of time; 2) If a warehouse is closed, it will not be available for reopening before a certain length of time. Even though these constraints are problem-specific, they can be useful in other applications. Further discussion on this topic can be found in Section 4.3.

The handling of safety stock, covered in section 4.4, is another novel aspect of this article. Most previous works that include this feature involve nonlinear models, which are much harder to solve (You and Grossmann, 2008; de Bittencourt et al., 2015). We propose a piecewise linear formulation that implicitly considers demand variability and the risk-pooling effect (Eppen, 1979).

With all the complicating issues considered, obtaining the optimal solution becomes a challenging task.

To solve larger problems, additional modeling schemes are considered. Valid inequalities are proposed to tighten the formulation and to reduce the solution time (section 5). Finally, simplifying formulations are derived based on reasonable assumptions (section 6). These formulations have a large impact on reducing the solution time with only a small increase in the total cost.

2 Literature Review

The unique mix of features included in the present work make it different to the previous articles published on facility location problems. The extensive literature in the area is covered in the comprehensive reviews by Owen and Daskin (1998), Klose and Drexl (2005), Melo et al. (2009) and Farahani et al. (2013). Owen and Daskin (1998) stress the difficulty of the problem pointing out that most of the resulting models are NP-Hard. They also classify the problems in static and deterministic, dynamic, and stochastic. In another review by Klose and Drexl (2005), besides analyzing special types of problems, the continuous and network location models, they classify mixed-integer models into several categories: network topology, objectives, capacitated or uncapacitated, number of stages, single and multi-commodity, dynamic and static, and deterministic and stochastic models. Melo et al. (2009) review 139 articles, and present classifications in terms of complementary decisions considered, solution strategy, network structure and application. Farahani et al. (2013) present another review to the subject, focusing on hub location problems.

The relevance of discrete transportation costs in supply chain design was pointed out by Gupta (1992). However, as it can be observed in the recent review by Bravo and Vidal (2013), focused on the freight cost function, most authors have neglected this issue. Discrete costs mean that the transportation cost is fixed per each truck or container, whether the unit is full or not. The total cost is then a piecewise constant function of the transported amount. Very few articles have taken into account this situation. Park and Hong (2009) considered a set of trucks and binary variables to indicate if the vehicle is used in a defined route and solved it with a Genetic Algorithm (Fogel, 1966). Manzini and Bindi (2009), in their strategic model III, consider integer variables for the transportation containers and the corresponding cost is expressed per container. Brahimi and Khan (2014), in a model for a multinational oil company supply chain, also consider integer variables to represent the number of trucks in each link. They solve a small scale single-product problem. Quttineh and Lidestam (2014) develop a planning model with boat and truck transportation using integer variables to account for the number of boats or trucks that are used in a determined link and period. They solve the problem using a rolling-horizon technique. However, they also attempted to solve a full scale model with 366 periods. After 48 hours of computation the optimality gap was still 3%. A different approach was taken by Gao et al. (2010) for a supply chain design problem with freight cost discounts in which the unit transportation cost is a piecewise function of the transported amount. The disjunction is reformulated using the convex-hull (Balas, 1998) and solved using Benders decomposition (Geoffrion, 1972). As reported in the literature, there are several alternatives to represent discrete freight costs. However, no previous work has done a critical review to identify the most efficient formulation among the alternatives. In this work we compare 4 alternatives of modeling the discrete transportation costs and identify the most efficient one (minimum solution time) for the current application.

Demand uncertainty is present in most supply chains. In order to hedge against stock-outs a safety stock is kept at warehouses. When the demand is assumed to be have a normal distribution, the magnitude of this inventory is given by the product of the demand variance (σ), the service level (z), and the square root of the replenishment lead time (L). Eppen (1979) showed that when more customers are served by the same warehouse, unexpected increases in the demand of one customer are offset by the unexpected decrease in the demand of another customer. Thus the joint variance decreases. This is known as the "risk-pooling effect". Several authors have included these concepts in supply chain design and planning models. The first example and the derivation of the formulation is presented in Daskin et al. (2002). Other examples include You and Grossmann (2008) and Miranda and Garrido (2009). The downside of the formulations in the aforementioned articles is that they give rise to nonlinear constraints in the model, making it very difficult to solve (de Bittencourt et al., 2015). This has motivated attempts of linearizing the constraints in order to keep the model as an MILP (Diabat and Theodorou, 2015). Unlike the previous works reported in the literature, we model the safety stock with risk-pooling effect using piecewise linear functions.

3 Problem Description

Given a set of plants producing a specified number of products, it is required to determine the location, number and size of warehouses to serve several customers in a region. The goal is to minimize the transportation and inventory costs.

Additionally to the given network topology, the problem has the following considerations and assumptions:

1. Not all plants produce all products. The allowed lanes in the network are defined by a set of feasible combinations of origin-destination-product.

2. At each link (plant-warehouse, warehouse-warehouse or warehouse-customer), there are several transportation modes that are available. The freight cost is a discrete amount defined by the number of transportation units used.

3. A transportation mode is defined by the capacity of the unit. For example, 10 ton trucks, 15 ton trucks and 20 ton oceanic containers each are considered different transportation modes. There is no limit in the number of available units.

4. Since the warehouses are contracted to third party logistics companies, there are minimum contract length restrictions, and when a contract is terminated, there is a minimum waiting period before a contract can be renewed.

5. The inventory flow cost is given by an inbound and outbound fee per unit of material. When the incoming or outgoing amount exceeds a defined threshold, a penalty must be paid by charging a larger fee for the amount that is exceeded.

6. The planing horizon is divided into discrete monthly periods.

7. A monthly demand forecast is available and assumed to be deterministic.

8. The warehouses must be selected among a set of previously defined potential locations.

Figure 1 depicts the problem and also indicates the main nomenclature used in the paper.

4 Optimization Model

The multi-period facility location problem serves as a basis for developing the optimization model. From this starting point, the modeling of each complicating issue will be addressed in detail to yield the final model.

4.1 The Uncapacitated Facility Location Problem

The uncapacitated facility location model is the core optimization formulation to solve supply chain design problems. It can be formulated with the following general model:

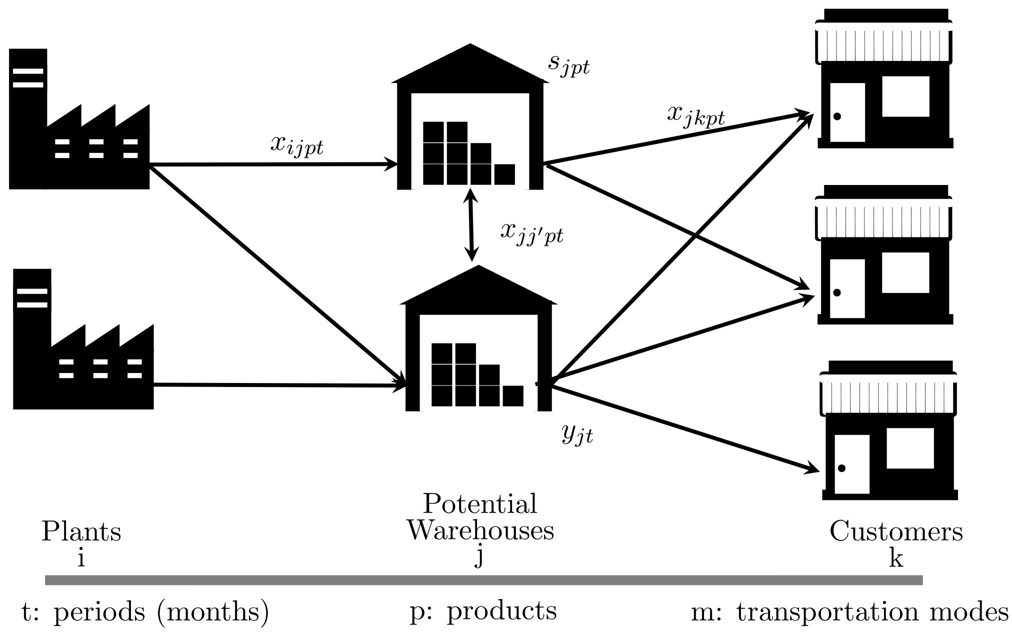


Figure 1. Network structure of supply chain

Min Fixed Cost + Inventory Cost + Production Cost + Transportation Cost

s.t. Demand satisfaction

Inventory balance

Conditional warehouse upper bound

where, the inventory costs include the flow costs and the warehouse renting costs. The conditional warehouse upper bound constraint serves to indicate when a warehouse is in use. A simple version is given by Equations (1a)-(1e) in which a single product is considered.

$$Min \quad \sum_t \left(\sum_j CF_j y_{j,t} + \sum_j HC_j s_{jt} + \sum_i \sum_j (PC_i + CT_{ij}) x_{ijt} + \sum_j \sum_k CT_{jk} x_{jkt} \right) \tag{1a}$$

$$s.t. \quad \sum_j x_{jkt} = D_{kt} \qquad \forall k, t \tag{1b}$$

$$s_{jt} = s_{jt-1} + \sum_i x_{ijt} - \sum_k x_{jkt} \qquad \forall j, t \tag{1c}$$

$$s_{jt} + \sum_k x_{jkt} \leq M y_{jt} \qquad \forall j, t \tag{1d}$$

$$x, s \geq 0, y \in \{0, 1\} \tag{1e}$$

Indices (i,j,k) denote the plant i ,warehouse j and customer k, respectively. x represents flow, s represents stock and y is the binary variable indicating the potential use of a warehouse in a given time period (see Figure 1). We initially consider, a single-commodity problem without shipments between warehouses. These features will be added later in the paper. Note that in the above model, the transportation cost is a linear function of the amount transported. This is the most common freight cost function used in the articles reviewed by Bravo and Vidal (2013). We will replace that function with the appropriate discrete representation after determining the most efficient formulation.

4.2 Discrete Transportation Costs

One of the most distinctive characteristics of the problem is that the transportation costs are discrete. The cost for each unit is fixed, no matter if the unit is full or half-full. At each time period and network link, the number of trucks or containers to be used must be selected. The units determined are then used to correctly account for the transportation costs.

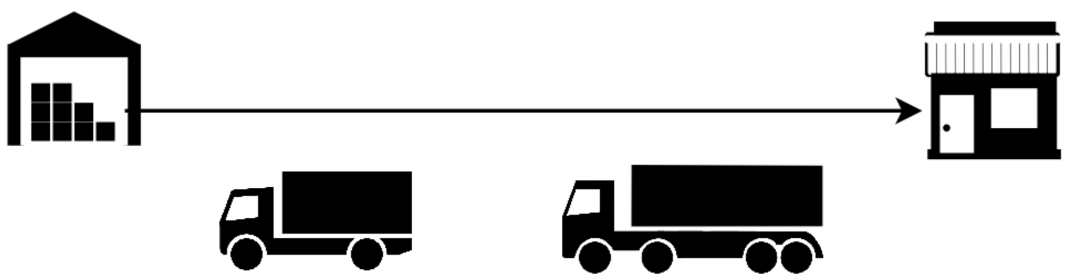


Figure 2. At a given time period and transportation link the number of units of each type must be determined

As reported in the literature, there are several alternatives to represent discrete freight costs. However, no previous work has done a critical review to identify the most efficient formulation. In this work we compare 4 alternatives of modeling the discrete transportation costs.

In the first alternative, given by Equation 2, integer variables are defined to compute the number of transportation units of each mode m used in a given link (warehouse j to customer k, for example) at a given time period t (u_{jkmt}), where $TCap_m$ is the capacity of transportation mode m. The inequality states that the transported capacity, given by the right hand side, must exceed the selected amount to be transported. In the following, to clarify the notation we would refer to the transported amount with an auxiliary variable v_{jkt}, defined as $v_{jkt} := \sum_p x_{jkpt}$.

$$v_{jkt} \leq \sum_{m \in M_{jk}} TCap_m \, u_{jkmt} \qquad \forall j,k,t \tag{2}$$

The second alternative is based on a piecewise constant freight cost function. For example, consider the case when 10 and 3 ton trucks are available and the cost of using a truck is \$400 and \$500, respectively. If the amount transported is between 0.1 and 3 tons, we would use one small truck and the freight cost would be \$400. If the amount is anything greater than 3 tons and less or equal than 10 tons, we would use one large truck and the cost would be \$500. If the amount is between 10 and 13 tons we would use a truck of each kind with a freight cost of \$900, and so on. The key concept is that for every amount there is a combination of trucks yielding the lowest cost, and the combination is the same in an interval for the amount. The cost function for the example is represented in Figure 3.

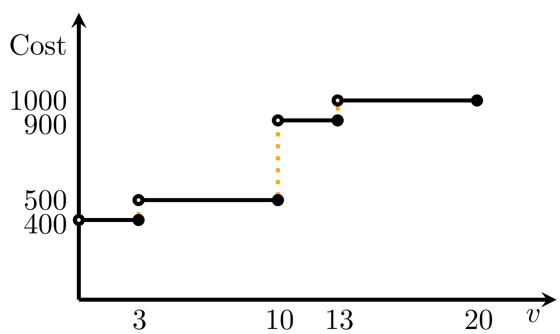


Figure 3. Piecewise constant cost function for the example

There are multiple ways of representing these functions (Padberg, 2000; Vielma and Nemhauser, 2011). For this work we have chosen to represent the function using Special Ordered Sets of Type II (SOS2) (Beale and Tomlin, 1970). Besides being ordered, the variables meet an additional adjacency condition: no more than two variables can be non-zero at a time. The non-zero variables must be adjacent. Variables of these kind are set to each of the n breakpoints of the stepwise function are used to interpolate and obtain the cost (Figure 4). Equations (3b)–(3d) represent the constraints that would be part of the model when selecting this alternative.

$$v_{jkt} = \sum_{n} PX_{jktn} \lambda_{jktn} \qquad \forall j,k,t \tag{3a}$$

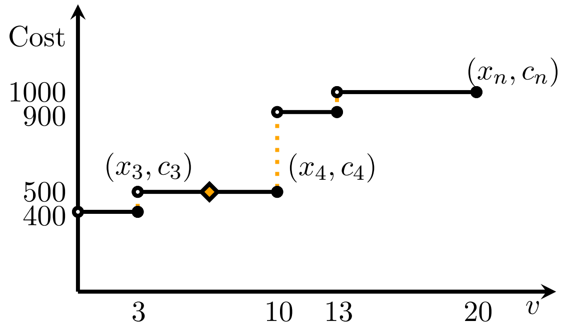


Figure 4. Piecewise constant cost function for the example

$$Cost = \sum_n PC_{jktn}\lambda_{jktn} \qquad \forall j,k,t \tag{3b}$$

$$\sum_n \lambda_{jktn} = 1 \qquad \forall j,k,t \tag{3c}$$

$$\lambda_{jktn} \in SOS2 \qquad \forall j,k,t,n \tag{3d}$$

where λ_{jktn} are the SOS2 variables, and (PX_{jtkn}, PC_{jktn}) are the breakpoints of the piecewise function for transported amount vs. cost.

The third and fourth alternatives are based on the disjunctive nature of the piecewise constant function. The transported amount can only be in one of the defined intervals of the cost function from (Figure 5). This leads to the disjunction from Equation 4.

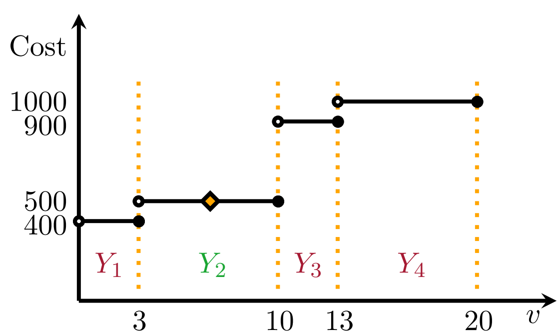


Figure 5. Piecewise constant cost function for the example

$$\bigvee_{n \in N} \begin{bmatrix} Y_{jktn} \\ L_{jkn} \leq v_{jkt} \leq U_{jkn} \\ FreightCost = C_{jkn} \end{bmatrix} \qquad \forall j,k,t \tag{4}$$

The Boolean variables Y_{jktn} take a value True if the amount v_{jkt} lies in the interval $[L_{jkn}, U_{jkn}]$. Raman and Grossmann (1994) describe a series of modeling techniques to deal with this kind of expressions in mathematical programming. One of them is known as the Big-M reformulation given by Equations (5a)–(5c).

$$L_{jk} y_{jktn} \leq v_{jkt} \leq U_{jkn} + M(1 - y_{jktn}) \qquad \forall j,k,t \qquad (5a)$$

$$\sum_n y_{jktn} = 1 \qquad \forall j,k,t \qquad (5b)$$

$$y_{jktn} \in \{0,1\} \qquad (5c)$$

where M is a big number and y_{jktn} are binary variables corresponding to each boolean variable Y_{jktn}.

Another way of reformulating the disjunction from Equation 4 is using the Convex Hull (Balas, 1998) given by Equations (6a)–(6d) where v_{jktn} are disaggregated variables.

$$v_{jkt} = \sum_n v_{jktn} \qquad \forall j,k,t \qquad (6a)$$

$$L_{jkn} y_{jktn} \leq v_{jktn} \leq U_{jkn} y_{jktn} \qquad \forall j,k,t,n \qquad (6b)$$

$$\sum_n y_{jktn} = 1 \qquad \forall j,k,t \qquad (6c)$$

$$y_{jktn} \in \{0,1\} \qquad \forall j,k,t,n \qquad (6d)$$

The four alternatives were compared for different model sizes for a simplified instance of the problem. The results are presented in Table 1. The values indicate solution times for each instance. When the 10 minutes time limit was reached the optimality gap is also reported. NF indicates that no feasible solution was found after the time limit was reached. The 10-minute limit was chosen because the instances from the experiment are quite small compared to the actual problem size intended to solve.

Table 1. Comparison of solution time. For all instances the number of plants and warehouses is 3 and 6 respectively

Instance	Customers	Products	Periods	Solution Time (s)			
				Integer Units	SOS2	Convex Hull	Big M
T6C4P1	4	1	6	1	4	180	60
T6C4P5	4	5	6	20	600(8.7%)	600(0.6%)	600(81%)
T12C8P5	8	5	12	60	600(12.6%)	NF	NF
T36C8P5	8	5	36	120	NF	NF	NF

As summarized in Table 1, the model with integer variables (Equation 2) was able to solve all the instances in under 2 minutes. Therefore, it is selected as the most efficient alternative to model discrete freight costs for the current application. A possible explanation for this result is the small number of variables and constraints of the model with integer variables compared to the other alternative models (Table 2), in a problem with 3 plants, 6 warehouses, 8 customers, 5 products, 36 time periods, and 4 modes of transportation. Furthermore, the LP relaxations for the convex hull and the integer variables formulations are very similar and tighter than the other alternatives.

Table 2. Model size comparison for modeling alternatives for transportation costs

	Continuous Variables	Discrete Variables	Constraints	T6C4P5 LP relaxation
Integer Variables	34280	8532	8000	33%
BigM	483344	453852	919736	96%
Convex Hull	936944	453852	924272	35%
SOS2	936944	252	17072	50%

4.3 Dynamic Contracting Policies

When the warehousing service is outsourced contracting must be done for continuous periods of time, and when a contract is finished it cannot be renewed right away. This restriction avoids the generation of short gaps in the use of a warehouse, which are difficult to fill with another customer. Figure 6 illustrates this situation.

	1	2	3	4	5	6	7	8	9	10	11	12
Not Allowed												
Allowed												

Figure 6. The second row is preferred because the warehouse is contracted in consecutive periods with few gaps

To enforce these restrictions, a minimum contracting length L and a minimum waiting period for contract renewal W are defined. The binary variable y_{jt} represents whether a warehouse j is used in period t or not. New binary variables y^s_{jt} and y^f_{jt} to indicate when a contract is started and finished (Mitra et al., 2012), respectively, are also defined. With these elements, Equations (7a)–(7d) are added to the model. The reader may notice that Equations (7b) and (7d) are surrogates of the constraints derived for a single time period. Our experiments show that there is no difference in the solution time between using the previous constraints or the disaggregated versions. The derivation of these constraints is presented in Appendix A.

$$-y_{jt} + y_{jt-1} + y^s_{jt} \geq 0 \qquad \forall j, t\ , t > 1 \tag{7a}$$

$$\sum_{\tau=t}^{t+L-1} y_{j\tau} \geq L y^s_{jt} \qquad \forall j, t\ , t + L - 1 \leq |T| \tag{7b}$$

$$-y_{jt} + y_{jt+1} + y^f_{jt} \geq 0 \qquad \forall j, t\ , t < |T| \tag{7c}$$

$$\sum_{\tau=t+1}^{t+W} y_{j\tau} + W y^f_{jt} \leq W \qquad \forall j, t\ , t + W \leq |T| \tag{7d}$$

If a warehouse j is used in a given period t and not in the previous period $(t-1)$, y^s_{jt} is set to one (Equation 7a), indicating a contract is started in period t. This implies that y_{jt} must be set to one for the next L-1 periods (Equation 7b). A similar logic is used for the case when a contract is finished. Note that Equations 7b and 7d can be disaggregated yielding a larger number of constraints but a tighter formulation. Our experiments indicated that there is no difference in the solution time between the two options.

4.4 Safety Stock with Risk-Pooling effect

When the demand is assumed to have a normal distribution, the safety stock can be expressed by Equation (8) ((Daskin et al., 2002)).

$$ss = \sigma z \sqrt{L} \tag{8}$$

To represent the safety stock with risk-pooling effect using only linear constraints we need to analyze this equation. First, for a given service level (z) and lead time (L), the safety stock is proportional to the absolute variance (σ), which is also proportional to the demand. Additionally, to account for the risk-pooling effect, the proportionality constant must decrease with the number of customers served, which is indirectly also represented by the throughput. The throughput can be measured as the amount entering or leaving the warehouse j. For this problem, the first (incoming amount from plants i) is more convenient because it allows for warehouses to close at any given time period without leaving safety stock. The safety stock can then be approximated by the following piecewise linear function defined over n intervals.

$$\sum_i x_{ijpt} = \sum_n \lambda_{jptn} SSx_n \qquad \forall j,p,t \tag{9a}$$

$$ss_{jpt} = \sum_n \lambda_{jptn} SSy_n \qquad \forall j,p,t \tag{9b}$$

$$\sum_n \lambda_{jpn} = 1 \qquad \forall j,p \tag{9c}$$

$$0 \leq \lambda_{jpn} \leq 1, \in SOS2 \qquad \forall j,p,n \tag{9d}$$

where (SSx, SSy) are the breaking points of the piecewise linear function(Figure 7).

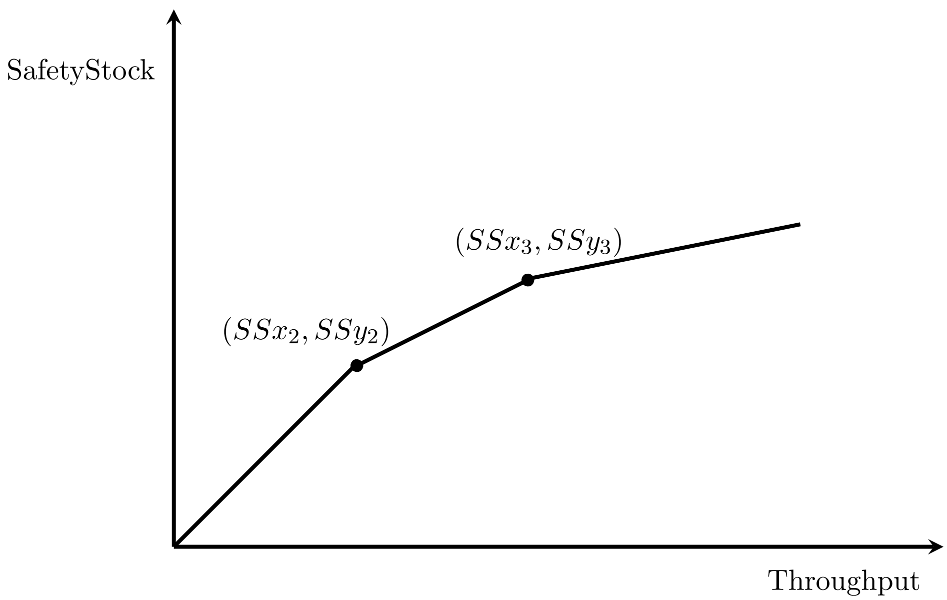


Figure 7. Safety stock piecewise linear function

4.5 Inventory Flow Costing

The inventory costing is quite atypical. It is given by a per-unit inbound and outbound fee, with a penalty for inbound or outbound amounts exceeding defined thresholds. None of the articles reviewed consider this special case. This costing structure allows warehousing providers to offer different business models. Low per unit fees and large fixed cost, or the opposite.

In order to efficiently model this situation, continuous variables to account for incoming (xwi_{jth}) and outgoing (xwo_{jth}) product at a warehouse must be defined. The variables are also indexed by the level h to distinguish between amounts below and above the penalty threshold. These variables are only used for costing purposes; they are not used to represent any decision. Figure 8 illustrates this case. The upper bound for the low volume level variable must be set to the threshold (TI_j, TO_j).

$$\sum_i \sum_p x_{ijpt} = xwi_{jt,low} + xwi_{jt,high} \qquad \forall j,t \tag{10}$$

$$xwi_{jt,low} \leq TI_j \qquad \forall j,t \tag{11}$$

$$\sum_k \sum_p x_{jkpt} = xwo_{jt,low} + xwo_{jt,high} \qquad \forall j,t \tag{12}$$

$$xwo_{jt,low} \leq TO_j \qquad \forall j,t \tag{13}$$

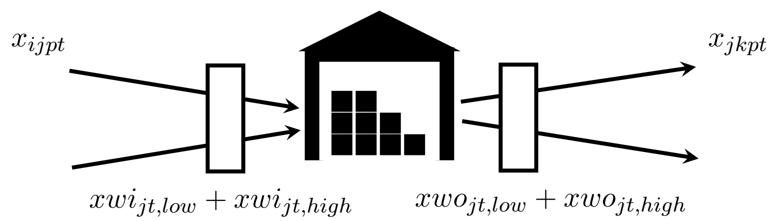


Figure 8. Cargo consolidation variables

4.6 Optimization model

Considering all the elements analyzed in the previous section the resulting optimization model can be formulated as follows:

Sets

- i Plants
- j Warehouses
- k Customers
- p Products
- t Time periods
- m Modes of transportation
- h High/Low storage volume
- n Safety stock interpolation points

Parameters

D_{kpt}	Demand of product p in period t at customer k
$TCap_m$	Capacity of a unit of transportation mode m
CT_{ijm}	Transportation cost from plant i to warehouse j using mode m [$/truck] or [$/container]
$CT_{jj'm}$	Transportation cost from warehouse j to warehouse j' using mode m [$/truck] or [$/container]
CT_{jkm}	Transportation cost from warehouse j to customer k using mode m [$/truck] or [$/container]
CI_{jh}	Inbound inventory fee at warehouse j per volume level h [$/ton]
CO_{jh}	Outbound inventory fee at warehouse j per volume level h [$/ton]
TI_j	Inbound threshold level at warehouse j [ton]
TO_j	Outbound threshold level at warehouse j [ton]
CWF_j	Monthly fixed cost for renting warehouse j [$]
CWS_j	Variable cost for renting warehouse j [$/$ton$]
PC_{ip}	Production cost of product p at plant i
SSx_{jpn}	Safety stock function breaking point
SSy_{jpn}	Safety stock function breaking point
$PCap_i$	Plant i capacity
M	Big number
L	Minimum warehouse contract length
W	Minimum wait for warehouse contract renewal

Continuous Variables

$x_{ijpt}, x_{jj'pt}, x_{jkpt}$	Amount of product p shipped in each link in period t
s_{jpt}	Inventory of product p in warehouse j at the end of period t
ss_{jpt}	Safety stock of product p in warehouse j at the end of period t
xwi_{jth}	Incoming quantity to warehouse j of volume level h in period t
xwo_{jth}	Incoming quantity to warehouse j of volume level h in period t
$wcap_j$	Capacity of warehouse j
$\lambda_{j,p,t,n}$	Interpolating coefficient for safety stock function

Discrete Variables

y_{jt}	Binary variable to indicate if the warehouse j is used in period t
y^s_{jt}	Binary variable to indicate if a contract with warehouse j is started in period t
y^f_{jt}	Binary variable to indicate if a contract with the warehouse j is finished in period t
u_{ijmt}	Units of transportation mode m used from plant m to warehouse j in period t
$u_{jj'mt}$	Units of transportation mode m used from warehouse j to another warehouse j' in period t
u_{jkmt}	Units of transportation mode m used from warehouse j to customer k in period t

Constraints

Demand satisfaction,

$$\sum_j x_{jkpt} = D_{kpt} \qquad \forall k,p,t \qquad (14)$$

Inventory balance,

$$s_{jpt} + ss_{jpt} = s_{jpt-1} + ss_{jpt-1} + \sum_i x_{ijpt} + \sum_{j'} x_{j'jpt} - \sum_{j'} x_{jj'pt} - \sum_k x_{jkpt} \qquad \forall j,p,t \qquad (15)$$

Warehouse upper bound,

$$s_{jpt} + \sum_k x_{jkpt} + \sum_{j'} x_{jj'pt} \leq M\ y_{jt} \qquad \forall j,p,t \qquad (16)$$

Plant capacity,

$$\sum_j \sum_p x_{ijpt} \leq PCap_i \qquad \forall i,t \qquad (17)$$

Transportation units,

$$\sum_p x_{ijpt} \leq \sum_m TCap_m\ u_{ijmt} \qquad \forall i,j,t \qquad (18)$$

$$\sum_p x_{jj'pt} \leq \sum_m TCap_m\ u_{jj'mt} \qquad \forall j,j',t \qquad (19)$$

$$\sum_p x_{jkpt} \leq \sum_m TCap_m\ u_{jkmt} \qquad \forall j,k,t \qquad (20)$$

Inventory Flow costing,

$$\sum_i \sum_p x_{ijpt} + \sum_{j'} \sum_p x_{j'jpt} = \sum_h xwi_{jth} \qquad \forall j,t \qquad (21)$$

$$xwi_{jt,low} \leq TI_j \qquad \forall j,t \qquad (22)$$

$$\sum_k \sum_p x_{jkpt} + \sum_{j'} \sum_p x_{jj'pt} = \sum_h xwo_{jth} \qquad \forall j,t \qquad (23)$$

$$xwo_{jt,low} \leq TO_j \qquad \forall j,t \qquad (24)$$

Warehouse capacity,

$$wcap_j \geq \sum_p s_{jpt} + \sum_h xwi_{jht} \qquad \forall j,t \qquad (25)$$

Warehouse contracting policies,

$$-y_{jt} + y_{jt-1} + y^s_{jt} \geq 0 \qquad \forall j,t \qquad (26)$$

$$\sum_{\tau=t}^{t+L-1} y_{j\tau} \geq L y^s_{jt} \qquad \forall j,t \qquad (27)$$

$$-y_{jt} + y_{jt+1} + y^f_{jt} \geq 0 \qquad \forall j,t \qquad (28)$$

$$\sum_{\tau=t+1}^{t+W} y_{j\tau} + W y^f_{jt} \leq W \qquad \forall j,t \qquad (29)$$

Safety stock with risk-pooling,

$$\sum_i x_{ijpt} = \sum_n \lambda_{jptn} SSx_{jpn} \qquad \forall j,p,t \qquad (30)$$

$$ss_{jpt} = \sum_n \lambda_{jptn} SSy_{jpn} \qquad \forall j,p,t \qquad (31)$$

$$\sum_n \lambda_{jptn} = 1 \qquad \forall j,p,t \qquad (32)$$

Objective Function

$$Min \quad TotalCost = WhCost + TranspCost + ProdCost + InvCost \qquad (33)$$

$$WhCost = \sum_j \sum_t y_{jt} CWF_j + \sum_j wcap_j CWV_j \qquad (34)$$

$$TranspCost = \sum_m \sum_t \left(\sum_i \sum_j u_{ijtm} CT_{ijm} + \sum_j \sum_{j'} u_{jj'tm} CT_{jj'm} + \sum_j \sum_k u_{jktm} CT_{jkm} \right) \qquad (35)$$

$$ProdCost = \sum_i \sum_j \sum_p \sum_t PC_{ip} x_{ijpt} \qquad (36)$$

$$InvCost = \sum_j \sum_h \sum_t (xwi_{jht} CI_{jh} + xwo_{jht} CO_{jh}) \qquad (37)$$

Variable Domains

$$x_{ijpt}, x_{jj'pt}, x_{jkpt}, s_{jpt}, ss_{jpt}, xwi_{jth}, xwo_{jth}, wcap_j \geq 0 \tag{38}$$

$$u_{ijtm}, u_{jj'tm}, u_{jktm} \in \mathbb{Z}_0^+ \tag{39}$$

$$y_{jt}, y_{jt}^s, y_{jt}^f \in \{1, 0\} \tag{40}$$

$$\lambda_{jptn} \in SOS2 \tag{41}$$

5 Tightening Constraints

The various features considered by the model make it more realistic but at the same time harder to solve. This is why additional effort needs to be made in the modeling for solving larger instances. The first alternative explored is to include valid inequalities in the formulation that are not strictly required to obtain the optimal solution, but contribute to strengthening the relaxation, and thus, potentially solve the problem faster. Four families of tightening constraints were studied. However, only one of them resulted in a speed up in the solution time. They are presented in Equations 42a–42c.

$$u_{ijtm} \leq \frac{1}{TCap_m} \sum_p x_{ijpt} + 1 \qquad \forall i, j, t, m \tag{42a}$$

$$u_{jj'tm} \leq \frac{1}{TCap_m} \sum_p x_{jj'pt} + 1 \qquad \forall j, j', t, m \tag{42b}$$

$$u_{jktm} \leq \frac{1}{TCap_m} \sum_p x_{jkpt} + 1 \qquad \forall j, k, t, m \tag{42c}$$

Constraints from Equations (42a)–(42c), provide a tight upper bound for the transportation units. They indicate that the number of transportation units used in a given link at specific time periods will be at most the number of units that would be used if that transportation mode is unique. For example, assume 12 tons of material need to be transported, and trucks of 3 and 10 tons capacity are available. The maximum number of 3-ton trucks used would be 4, and the maximum number of 10-ton trucks used would be 2. Even though, it is very likely that the minimum cost alternative would be to use one of each size.

6 Simplified Formulations

Another strategy to decrease the solution time is to make reasonable assumptions to simplify the MILP model to obtain an approximate solution. Two formulations are proposed based on assumptions of customers service policies.

The first simplified formulation assumes that a given customer receives a given product from a single warehouse. In the following, we will refer to this formulation as JKP, because only one of the combinations warehouse-customer-product is allowed (single-sourcing). For example, if a customer demands products A and B, it could receive product A from one warehouse (W1), and product B from another warehouse (W2), but it could not receive the same product from two separate warehouses. The assumption is reasonable because products supply tend to follow minimum plant-warehouse-customer cost routes. The deviation from this assignment only occurs when limitations of capacity are reached. On the other hand, the network design is primarily driven by transportation costs on the warehouse-customer side. The modeling simplification is that the variable that represents the flow between a warehouse j and a customer k for a given product p in time period t, x_{jkpt}, can be replaced by the product between a new binary variable, z_{jkp}, and the Demand, D_{kpt}. An additional constraint is needed to ensure the combination warehouse-customer-product is unique. The variable z_{jkp} takes a value of one if warehouse j supplies product p to customer k

$$\sum_j z_{jkp} = 1 \qquad \forall k, p \tag{43}$$

With this new formulation (JKP), Equations (15,16, 20, 23) are replaced by Equations (44-47). Equation (14) is no longer needed.

$$s_{jpt} + ss_{jpt} = s_{jpt-1} + ss_{jpt-1} + \sum_i x_{ijpt} + \sum_{j'} x_{j'jpt} - \sum_{j'} x_{jj'pt} - \sum_k z_{jkp} D_{kpt} \qquad \forall j, p, t \tag{44}$$

$$s_{jpt} + \sum_k z_{jkp} D_{kpt} + \sum_{j'} x_{jj'pt} \leq \mathbb{U}\, y_{jt} \qquad \forall j, p, t \tag{45}$$

$$\sum_p z_{jkp} D_{kpt} \leq \sum_m TCap_m\, u_{jkmt} \qquad \forall j, k, t \tag{46}$$

$$\sum_k \sum_p z_{jkp} D_{kpt} + \sum_{j'} \sum_p x_{jj'pt} = \sum_h xwo_{jth} \qquad \forall j, t \tag{47}$$

It is important to observe that with the JKP formulation, a large number of continuous variables, x_{jkpt}, are replaced by smaller yet significant number of binary variables, z_{jkp}. Thus it is not straightforward to predict a decrease in solution time. However, our experiments, presented in section 7, show that the impact of the reformulation is indeed positive. The second observation is that since this model represents a restriction of the original model, the objective value provides a valid upper bound cost of the original problem. Our experiments have indicated that this bound is very tight and typically no more than 1% higher than the optimal cost.

Taking this idea further, we can also assume that a customer receives all its demanded products from a given warehouse. In this formulation (JK), the binary variable z_{jk} indicates this assignment. As before, the variable x_{jkpt} is replaced by $z_{jk}D_{kpt}$, but additionally the transportation units in the warehouse-customer links can be precalculated offline, eliminating the integer variable u_{jktm}. In this way, the number of binary variables added is much smaller than before. Furthermore, a large number of continuous and integer variables is eliminated, thereby reducing the model complexity. Since the assumption is even more restrictive, the resulting objective value yields an upper bound to both, the original problem and the JKP formulation. With this formulation Equations (15,16, 23) are replaced by Equations (48-50). Equations (14, 20) are no longer needed.

$$s_{jpt} + ss_{jpt} = s_{jpt-1} + ss_{jpt-1} + \sum_i x_{ijpt} + \sum_{j'} x_{j'jpt} - \sum_{j'} x_{jj'pt} - \sum_k z_{jk} D_{kpt} \qquad \forall j,p,t \qquad (48)$$

$$s_{jpt} + \sum_k z_{jk} D_{kpt} + \sum_{j'} x_{jj'pt} \leq \mathbb{U}\, y_{jt} \qquad \forall j,p,t \qquad (49)$$

$$\sum_k \sum_p z_{jk} D_{kpt} + \sum_{j'} \sum_p x_{jj'pt} = \sum_h xwo_{jth} \qquad \forall j,t \qquad (50)$$

As with the JKP formulation, a constraint is needed to ensure the combination warehouse-customer is unique.

$$\sum_j z_{jk} = 1 \qquad \forall k \qquad (51)$$

7 Case Studies

To illustrate the capability and performance of the proposed models, three case studies are presented in this section. All the instances considered are problems of different size, with the same features described in Section 3.

7.1 Small Example

Now we provide a small example, case study 1, representative of the full problem so it can be reproduced by the readers. The problem considers 2 transportation modes, 3 products (A,B,C), 2 plants, 2 warehouses and 2 customers. The planning horizon is 6 months, and the minimum contract duration and minimum renewal wait are both 3 months. Tables 5–8 provide the data required for the model.

Table 5. Plant data for small example

Plant	ProdCap [t/month]	ProdCost [$/t]		
		A	B	C
M1	2137	0.20	0.64	
M2	3674		0.47	0.52

Table 6. Warehouse data for small example

Warehouse	FixedCost	VarCost	Threshold	InboundCost	OutboundCost
WHP1	5906	0.1	0	0	0
WHP2	0	0.1	0	0.00779	0.00779

Table 7. Demand(t) for small problem

Periods	Customer 1			Customer 2		
	A	B	C	A	B	C
1	259.6			8.7	4.0	2.4
2	115.4		2.9	54.5	10.3	6.5
3	194.5			52.7	64.8	20.1
4	161.9			28.7	20.7	15.9
5	217.0		4.2	9.0	58.6	26.1
6	234.3			53.5	114.8	33.7

The optimal network with a cost of $213,149 is presented in Figure 9. The MILP model involves 698 variables, 156 discrete variables, and 536 constraints. It was solved in 0.24 seconds using Gurobi 6.5 (Gurobi Optimization, 2015) on GAMS 24.6.1 (GAMS Development Corporation, 2015).

Figure 10 presents the number of transportation units used in each time period. The model selects the largest trucks (32 ton) in most periods. However, as it can be seen in the points for period 5, this cannot be generalized since the selected units will be those who match better the transported amount.

Table 8. Transportation costs for small problem

Orig/Dest	24 ton truck				32 ton truck			
	CUS1	CUS2	WHP1	WHP2	CUS1	CUS2	WHP1	WHP2
MFG3			1.20	1.20			1.48	1.48
MFG5	0.97	1.55	1.17	1.17	1.26	1.83	1.45	1.45
WHP1	1.10	1.30		0.73	1.38	1.57		1.02
WHP2	1.12	1.27	0.73		1.40	1.55	1.02	

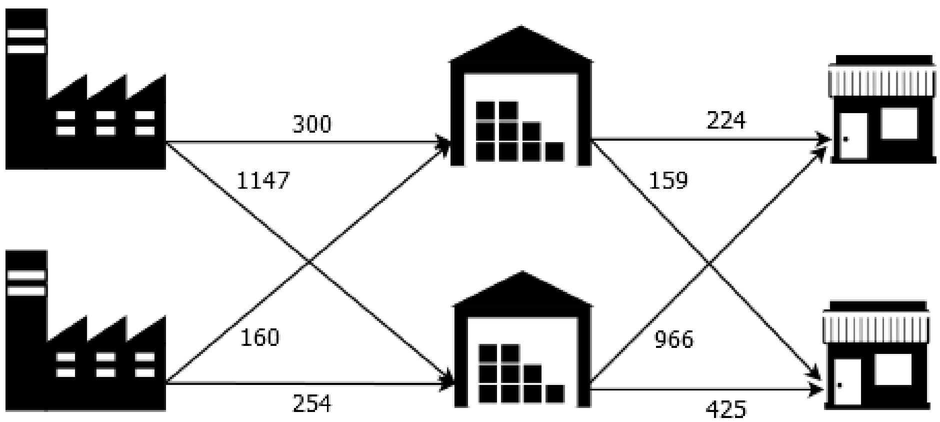


Figure 9. Total flows over 6 months for the optimal supply chain network from for case study 1

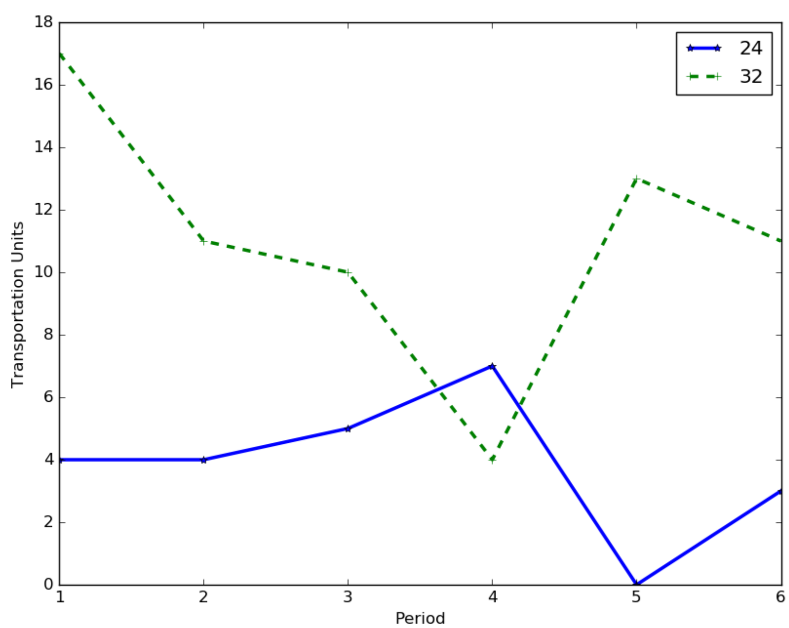
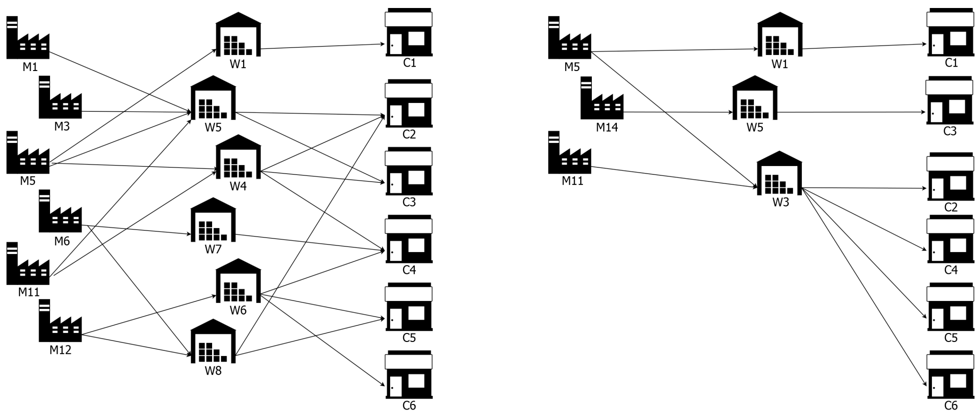


Figure 10. Number of transportation units used per time period in case study 1

7.2 The importance of considering discrete transportation costs

Including discrete transportation costs in a supply chain model makes the problem much harder to solve. On the other hand, considering continuous costs proportional to the transported quantity leads to sub-optimal solutions. To quantify the magnitude of the trade-off between these two approaches, consider case study 2 with 8 plants, 10 warehouses, 6 customers, 5 products, 24 time periods and 4 transportation modes.

The solution with continuous costs is obtained by relaxing the integrality of the transportation unit variables (u_{ijtm},$u_{jj'tm}$,u_{jktm}). Table 9 shows the results using continuous and discrete transportation costs. The resulting supply chain network structures are presented in Figure 11.



(a) Continuous costs formulation **(b)** Discrete costs formulation

Figure 11. Optimal network structures for Case Study 1

Table 9. Comparison of discrete transportation costs versus continuous proportional transportation costs in case study 2

Metric	Continuous Cost	Discrete Cost
Objective	84.1	262.4
Inventory(ton)	193	723
Shipments	807	417
Avg.Shipment Size	1.8	2.9
Constraints	20422	20422
Variables	54840	54840
Integer variables	0	21912
Binary variables	528	528
CPU(s)	53	10350

The continuous transportation cost formulation solves in at least one order of magnitude faster than the discrete cost formulation. However, it fails to correctly estimate the total costs, it does not allow to identify the transportation modes that are used (it selects the lowest unit cost mode), and obtains a suboptimal network. Another important observation is that the continuous costs formulation obtains a plan with frequent and small shipments, which is not desirable in practice. It fails to capture the benefits of economies of scale in transportation. For these reasons it is important to consider discrete freight costs.

7.3 Tightening constraints and simplified formulations

For case study 3, the effect of tightening constraints and the simplified formulations JKP and JK were evaluated in three instances of different size. The results are presented in Table 10

Table 10. Case study 3 objective values, solution times(s) and explored nodes for the original formulation (Orig), the original formulation including tightening constraints (Orig-t) and the simplified formulations JKP and JK

Instance	Size				Metric	Orig	Orig-t	JKP	JK
C10P10T12	Plants	14	Products	10	Obj	2798	2798	2798	2799
	Warehouses	18	Periods	12	CPU(s)	15	15	10	18
	Customers	10	Modes	14	Nodes	0	0	0	0
C10P10T24	Plants	14	Products	10	Obj	5422	5422	5422	5430
	Warehouses	18	Periods	24	CPU(s)	379	379	84	71
	Customers	10	Modes	14	Nodes	1469	1469	0	0
C15P15T36	Plants	14	Products	15	Obj	10316	10316	10373	10417
	Warehouses	18	Periods	36	CPU(s)	3667	3650	366	177
	Customers	15	Modes	14	Nodes	2506	2506	0	0

Table 11. Problem size for instance C15P15T36

	Orig	Orig-t	JKP	JK
Constraints	104717	150329	104717	104717
Variables	292759	292759	234834	212114
Integer variables	45378	45378	45378	22482
Binary variables	1332	1332	4475	1414

As shown in Table 10, the introduction of tightening constraints yields a negligible reduction in solution time for instances C10P10T24 and C15P15T36. There is no reduction in solution time by using the simplified formulations for the smallest instance, C10P10T12. However, up to 95% reductions are observed for the two larger problems.

The increase in the objective value of the JKP fomulation is very small (up to 0.55% higher), while the one at the JKP formulation has slightly higher value (up to 0.98%). Tables 12 and 13, and Figure 12 present details on the optimal solution of the instance C15P15T36.

Table 12. Optimal warehouse selection summary for instance C15P15T36. All amounts in tonnes

Warehouse	Capacity	Total Stock	Throughput
WH1	3350	17167	64991
WH3	287	1209	3763
WH6	760	2520	11004

Table 13. Transportation modes used in optimal solution for instance C15P15T36

Mode Capacity	Units
3.5	3
4.5	3
8	2
10	21
15	17
20	898
24	5
25	765
25	1028
32	3123
35	1

8 Conclusions and Future Work

In this paper we have addressed the optimal network design of a supply chain with seasonal demand as a facility location problem. The best formulation to address the distinctive characteristics of the supply chain under study was identified among several options and solved for 3 case studies, the larger of which involved 14 plants, 18 warehouses, 15 customers, 15 products, 36 time periods, and 14 modes of transportation.

The use of integer variables resulted in the most efficient formulation to address discrete transportation costs. As was observed in the results of Table 1, there are several ways to model the problem; the choice of a formulation can make the difference between finding the optimum and not being able to solve the problem. The safety stock was modeled using a piecewise linear approximation, and specific contracting policy constraints were derived from propositional logic. This is why, when facing an optimization problem it is critical to devote enough time to identify the most efficient formulation. Another lesson in terms of formulation is the variety of different alternatives reported in the literature. For a similar problem, several modeling alternatives can be found. However, clear consensus on which is the best among them is rarely found. Therefore, for every problem some degree of experimentation is required to determine the best formulation for the intended application.

All the features considered contribute to have more realistic models, especially when outsourcing

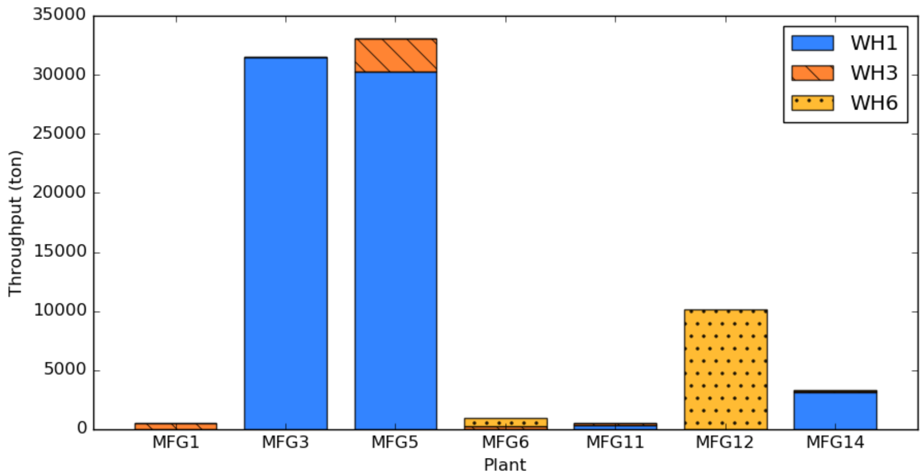


Figure 12. Optimal plant throughput summary for instance C15P15T36

logistic operations. But at the same time they pose a challenge in solving the optimization problems. The first steps towards solving larger problems are presented. Valid inequalities that help to tighten the relaxation were derived. It was shown that the simplified formulations JKP and JK help significantly to reduce the solution time, allowing to solve larger problems with a small impact in the objective value. Larger problems will require an additional effort in designing efficient algorithms employing decomposition techniques.

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References

Balas, E. . Disjunctive programming: Properties of the convex hull of feasible points. *Discrete Applied Mathematics*, 89(1):3–44, 1998.

Beale, E. M. L. and Tomlin, J. A. . Special facilities in a general mathematical programming system for non-convex problems using ordered sets of variables. *OR*, 69(99):447–454, 1970.

Brahimi, N. and Khan, S. a. . Warehouse location with production, inventory, and distribution decisions: A case study in the lube oil industry. *4or*, 12:175–197, 2014.

Bravo, J. J. and Vidal, C. J. . Freight transportation function in supply chain optimization models: A critical review of recent trends. *Expert Systems with Applications*, 40(17):6742–6757, 2013.

Cornuéjols, G. , Nemhauser, G. L. , and Wolsey, L. A. . The uncapacitated facility location problem. Technical report, DTIC Document, 1983.

Daskin, M. S. , Coullard, C. R. , and Shen, Z.-J. M. . An inventory-location model: Formulation, solution algorithm and computational results. *Annals of operations research*, 110(1-4):83–106, 2002.

Bittencourt, de G. C. , Ferreira Filho, V. J. M. , and Silva Leite, da L. S. B. . Inventory location problem with risk pooling effect: A performance analysis between different formulations. *Simposio Brasileiro de Pesquisa Operacional*, 2015.

Diabat, A. and Theodorou, E. . A location–inventory supply chain problem: Reformulation and piecewise linearization. *Computers & Industrial Engineering*, 90:381–389, 2015.

Eppen, G. D. . Effects of centralization on expected costs in a multi-location newsboy problem. *Management Science*, 25(5):498–501, 1979.

Farahani, R. Z. , Hekmatfar, M. , Arabani, A. B. , and Nikbakhsh, E. . Hub location problems: A review of models, classification, solution techniques, and applications. *Computers and Industrial Engineering*, 64(4):1096–1109, 2013.

Fogel, L. J. . *Artificial Intelligence Through Simulated Evolution*. John Wiley & Sons, 1966.

GAMS Development Corporation, . . General algebraic modeling system (gams) release 24.6.1, 2015. URL `http://www.gams.com/`.

Gao, D. , Jiao, W. , and Zhang, J. . Capacitated facility location problem with freight cost discount. *2010 7th International Conference on Service Systems and Service Management, Proceedings of ICSSSM' 10*, pages 249–253, 2010.

Geoffrion, A. M. . Generalized benders decomposition. *Journal of optimization theory and applications*, 10(4):237–260, 1972.

Gupta, O. K. . A lot-size model with discrete transportation costs. *Computers & industrial engineering*, 22(4):397–402, 1992.

Gurobi Optimization, . . Gurobi optimizer reference manual, 2015. URL `http://www.gurobi.com`.

Klose, A. and Drexl, A. . Facility location models for distribution system design. *European Journal of Operational Research*, 162:4–29, 2005.

Manzini, R. and Bindi, F. . Strategic design and operational management optimization of a multi stage physical distribution system. *Transportation Research Part E: Logistics and Transportation Review*, 45 (6):915–936, 2009.

Melo, M. T. , Nickel, S. , and Gama, Saldanha-da F. . Facility location and supply chain management–a review. *European journal of operational research*, 196(2):401–412, 2009.

Miranda, P. A. and Garrido, R. A. . Inventory service-level optimization within distribution network design problem. *International Journal of production economics*, 122(1):276–285, 2009.

Mitra, S. , Grossmann, I. E. , Pinto, J. M. , and Arora, N. . Optimal production planning under time-sensitive electricity prices for continuous power-intensive processes. *Computers and Chemical Engineering*, 38:171–184, 2012.

Owen, S. H. and Daskin, M. S. . Strategic facility location: A review. *European Journal of Operational Research*, 111:423–447, 1998.

Padberg, M. . Approximating separable nonlinear functions via mixed zero-one programs. *Operations Research Letters*, 27(1):1–5, 2000.

Park, Y.-B. and Hong, S.-C. . Integrated production and distribution planning for single-period inventory products. *International Journal of Computer Integrated Manufacturing*, 22(5):443–457, 2009.

Quttineh, N.-H. and Lidestam, H. . Using rolling horizon techniques in the planning process for a chemical process industry. In *18th International Working Seminars on Production Economics, Innsbruck, Austria, February 2014.*, pages 381–393, 2014.

Raman, R. and Grossmann, I. E. . Modelling and computational techniques for logic based integer programming. *Computers & Chemical Engineering*, 18(7):563–578, 1994.

Vielma, J. P. and Nemhauser, G. L. . Modeling disjunctive constraints with a logarithmic number of binary variables and constraints. *Mathematical Programming*, 128(1-2):49–72, 2011.

You, F. and Grossmann, I. E. . Mixed-integer nonlinear programming models and algorithms for large-scale supply chain design with stochastic inventory management. *Industrial & Engineering Chemistry Research*, 47(20):7802–7817, 2008.

Contents

Appendices

A Modeling of contracting policies

In this section we present the derivation of the formulation the Contracting Policy constraints. The constraints must enforce a minimum contract length, and, once a warehouse is closed, a minimum wait for the warehouse to be available for opening again. In order to correctly represent the situation, new binary variables need to be defined.

y_t^s Indicates that a contract has been started during period t

y_t^f Indicates that a contract has been finished at the end of period t

The variable y_t represents if the warehouse is used during the period t.

A.1 Contract start/finish variable setting constraints

To indicate that a contract has been started we check the warehouse is being used in the current period and not being used in the previous one.

$$y_t \wedge \neg y_{t-1} \implies y_t^s$$

The corresponding constraint is,

$$-y_t + y_{t-1} + y_t^s \geq 0 \tag{52}$$

To indicate that a contract is finished, we check that the warehouse is being used in the current period and not used in the next period,

$$y_t \wedge \neg y_{t+1} \implies y_t^f$$

The corresponding constraint is

$$-y_t + y_{t+1} + y_t^f \geq 0 \tag{53}$$

A.2 Minimum contract length constraints

When a contract is started the warehouse needs to be used in the next L periods. The logic expression describing this proposition is:

$$y_t^s \implies \bigwedge_{\tau=t}^{t+L-1} [y_\tau]$$

which can be represented as a linear constraint with the following steps:

$$\neg y_t^s \wedge \left[\bigwedge_{\tau=t}^{t+L-1} [y_\tau] \right] \tag{54}$$

$$\bigwedge_{\tau=t}^{t+L-1} [\neg y_t^s \wedge y_\tau] \tag{55}$$

$$1 - y_t^s + y_\tau \geq 1 \qquad \forall \tau \tag{56}$$

which can be aggregated as the surrogate,

$$\sum_{\tau=t}^{t+L-1} y_\tau \geq L y_t^s \tag{57}$$

A.3 Minimum waiting period for rehire constraints

When a contract is finished it cannot be renovated in the next W periods. The logic expression describing this proposition is:

$$y_t^f \implies \bigwedge_{\tau=t+1}^{t+W} [\neg y_\tau]$$

which can also be represented as a linear constraint.

$$\neg y_t^f \wedge \left[\bigwedge_{\tau=t}^{t+W} [\neg y_\tau] \right] \tag{58}$$

$$\bigwedge_{\tau=t}^{t+W} [\neg y_t^f \wedge \neg y_\tau] \tag{59}$$

$$1 - y_t^f + 1 - y_\tau \geq 1 \qquad \forall \tau \tag{60}$$

which can be aggregated as the surrogate,

$$\sum_{\tau=t+1}^{t+W} y_{\tau} + W y_t^f \leq W \tag{61}$$

Eq. (56) yields tighter constraints than the surrogate Eq. (57) but a much larger number of constraints. A similar situation is found between Eqs. (60) and (61)