

EFFICIENT FORMULATIONS FOR DYNAMIC WAREHOUSE LOCATION UNDER DISCRETE TRANSPORTATION COSTS

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Abstract

A Mixed-Integer Linear Programming model is proposed to determine the optimal number, location and capacity of the warehouses required to support a long-term forecast for a business with seasonal demand. Discrete transportation costs, dynamic warehouse contracting, and the handling of safety stock are the three main distinctive features of the problem. Four alternatives for addressing discrete transportation costs are compared. The most efficient formulation is obtained using integer variables to account for the number of units used of each transportation mode. Contracting policies constraints are derived to ensure warehouses are used for continuous periods. Safety stock with risk-pooling effect is considered using a piecewise-linear representation. To solve large-scale problems, tightening constraints, and simplified formulations are proposed. The simplified formulations are based on single-sourcing assumptions and yield near-optimal results with a large reduction in the solution time with a small increase in the total cost.

Keywords

Supply chain design, Facility location, Discrete transportation costs, Safety stock.

Introduction

Supply chains have become increasingly complex in recent years. Globalization has made a large number of new markets and sourcing options available. The response from many companies to this situation has been to focus on their core business, outsourcing the logistics and warehousing operations. Strategic decisions can have a large impact in the success of a company. This is why such decisions must be made using the best tools available.

A Mixed-Integer Linear Programming model that includes discrete transportation costs, dynamic warehousing contracting policies and safety stock with

risk-pooling effect (Eppen, 1979), is proposed in this paper. These features are especially important when the logistics and warehousing operations are outsourced. The goal of the proposed model is to determine the optimal number, location and size of warehouses in a supply chain for a business with seasonal demand. This is not a trivial task because decisions on production of each commodity, transportation mode selection, flow and inventory must be optimized simultaneously. The features considered make the model more realistic, but at the same time significantly harder to solve. This is why efficient model formulations

and solution strategies must be developed. This paper focuses on developing efficient formulations for the problem.

The paper is a contribution to the research in the facility location problem. The extensive literature in the area is covered in the comprehensive reviews by Owen and Daskin (1998), Klose and Drexl (2005), Melo et al. (2009), Farahani et al. (2013).

Discrete transportation costs are present in most supply chains. However, only a handful of problems consider this characteristic (Bravo and Vidal, 2013). Discrete costs mean that the transportation cost is fixed per each truck or container, whether the unit is full or not. The total cost is then a piecewise constant function of the transported amount. Park and Hong (2009) use an assignment problem approach. Another option is to use integer variables to represent the number of transportation units. This is presented by Manzini and Bindi (2009), Brahimi and Khan (2014), and Quttineh and Lidestam (2014). Gao et al. (2010) use a piecewise function to represent the transportation cost. In this work we compare 4 alternatives of modeling the discrete transportation costs and identify the most efficient one (minimum solution time) for the current application.

Another important feature considered is the warehouse contracting policies. Since the inventory storage service is supplied by an external company, constraints to ensure a continuous service must be enforced. Constraints derived from propositional logic impose two conditions: 1) once a warehouse is opened it must remain opened by at least a certain amount of time; 2) if a warehouse is closed, it will not be available for reopening before a certain amount of time.

The handling of safety stock is another novel aspect of this article. Daskin et al. (2002) consider safety stock with risk-pooling effect (Eppen, 1979), with a nonlinear formulation. You and Grossmann (2008), and Miranda and Garrido (2009) propose similar formulations. We propose a piecewise-linear formulation that implicitly considers demand variability and the risk-pooling effect.

With all the complicating issues included, obtaining the optimal solution becomes a challenging task. To solve larger problems tightening constraints and simplifying formulations are considered. These formulations have a big impact on the solution time, with only a small increase in the objective value.

Problem Description

Given a set of plants producing a specified number of products, it is required to determine the location, number and size of warehouses to serve several customers in a region. The goal is to minimize the transportation and inventory costs. A monthly demand forecast is available. Therefore, the planning horizon is divided in monthly periods. Figure 1 depicts the problem and also outlines the main nomenclature used in the paper.

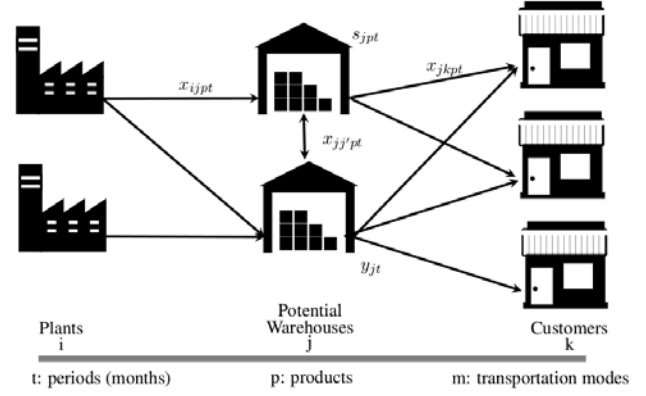


Figure 1: Network structure of supply chain

Optimization Model

The uncapacitated facility location model is the core optimization formulation to solve supply chain design problems. It can be formulated with the following general model:

$$\text{Min} \sum_{j,t} FC_j y_{jt} + \sum_{j,p,t} HC_j s_{jpt} + \sum_{j,j',p,t} CT_{jj'} x_{jj'pt} \quad (1)$$

$$+ \sum_{i,j,p,t} (PC_i + CT_{ij}) x_{ijpt} + \sum_{j,k,p,t} x_{jkp}$$

$$\sum_p x_{jkpt} = D_{kpt} \quad \forall k, p, t \quad (2)$$

$$s_{jpt} = s_{jpt-1} + \sum_i x_{ijpt} - \sum_k x_{jkpt} \quad \forall j, p, t \quad (3)$$

$$s_{jpt} + \sum_k x_{jkpt} \leq M y_{jt} \quad \forall j, p, t \quad (4)$$

$$x, s \geq 0, y \in \{0,1\} \quad (5)$$

Indices (i,j,k) denote the plant, warehouse and customer respectively. X represents flow, s represents stock and y is the binary variable indicating the use of a warehouse in a given period. FC , HC , CT and PC are the fixed, holding transportation and production costs, respectively. D represents the demand. Note that in the above model, the transportation cost is a linear function of the amount transported. We will replace that function with the appropriate discrete representation after determining the most efficient formulation.

Discrete Transportation Costs

In the first alternative considered, given by Eq. (6), integer variables are defined to compute the number of transportation units of each mode m used in a given link

(warehouse j to customer k , for example) at a given time period t (u_{jkm}). The inequality states that the transported capacity, given by the right hand side, must exceed the selected amount to be transported.

$$\sum_p x_{jkpt} \leq \sum_m TCap_m u_{jktm} \quad \forall j, k, t \quad (6)$$

The second alternative is based on direct interpolation on the cost function (Figure 2). The transportation cost is obtained interpolating with the transported quantity into the piecewise function, SOS2 variables are used to represent the function. Eq. (7)-(10) are the constraints included in this alternative.

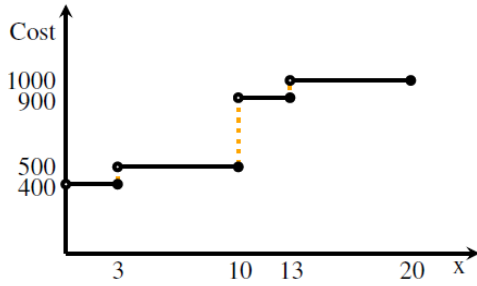


Figure 2. Piecewise cost function example

$$x_{jkpt} = \sum_n PX_{jkptn} \lambda_{jkptn} \quad \forall j, k, p, t \quad (7)$$

$$Cost_{jkpt} = \sum_n PC_{jkptn} \lambda_{jkptn} \quad \forall j, k, p, t \quad (8)$$

$$\sum_n \lambda_{jkptn} = 1 \quad \forall j, k, p, t \quad (9)$$

$$\lambda_{jkptn} \in SOS2 \quad (10)$$

where (PX_n, PC_n) are the breakpoints of the piecewise function

The third and fourth alternatives are based on the disjunctive nature of the piecewise constant function. The transported amount can only be in one of the defined intervals of the cost function from Figure 2. This leads to the disjunction from Eq. (11).

$$\bigvee_{n \in N} \left[\begin{array}{l} L_{jkn} \leq \sum_p x_{jkpt} \leq U_{jkn} \\ Cost_{jkt} = Cost_{jkn} \end{array} \right] \quad \forall j, k, t \quad (11)$$

The disjunction can be reformulated using the Big-M from Eq. (12)-(13) (Raman and Grossmann, 1994), or the Convex Hull from Eq. (14)-(15) (Balas, 1998). Eq (16)-(17) are common for both formulations.

$$L_{jkn} y_{jkt} \leq \sum_p x_{jkpt} \quad \forall j, k, t, n \quad (12)$$

$$\sum_p x_{jkpt} \leq U_{jkn} + M(1 - y_{jkt}) \quad \forall j, k, t, n \quad (13)$$

$$\sum_p x_{jkpt} = \sum_n f_{jkt} \quad \forall j, k, t, n \quad (14)$$

$$L_{jkn} y_{jkt} \leq f_{jkt} \leq U_{jkn} y_{jkt} \quad \forall j, k, t, n \quad (15)$$

$$\sum_n y_{jkt} = 1 \quad \forall j, k, t, n \quad (16)$$

$$y_{jkt} \in \{0,1\} \quad \forall j, k, t, n \quad (17)$$

The four alternatives were compared for different model sizes of the problem. The results are presented in Table 1. The values indicate the optimality gap after 10 min of run with a target of 0.5%. When a value of 0.5% is reported the model solved the problem in 10 minutes or less. NF indicates that no feasible solution was found after the time limit was reached. The instance code indicates the problem size. For example, T6C4P5 indicates 6 periods, 4 customers and 5 products. The 10-minute limit was chosen because the instances from the experiment are quite small compared to the size of the actual problem.

Table 1. Optimality gap after 10 min

Instance	Integer Variables	SOS2	BigM	Convex Hull
T6C4P1	0.5%	0.5%	0.5%	0.5%
T6C4P5	0.5%	8.7%	0.6%	81%
T12C8P5	0.5%	12.6%	NF	NF
T36C8P5	0.5%	NF	NF	NF

As summarized in Table 1, the model with integer variables (Eq. 6) was able to solve all the instances in less than 2 minutes. Therefore, it is selected as the most efficient alternative to model discrete freight costs for the current application. A possible explanation for this result is the small number of variables and constraints of the model with integer variables compared to the other alternative models. For each integer variable many extra variables and constraints are required in the other formulations.

Warehouse Contracting Policies

When the warehousing service is outsourced contracting must be done for continuous periods of time. When a contract is started, the warehouse must remain opened for at least a minimum number of periods. When a contract is finished it cannot be renewed right away, it must remain closed for at least a number of periods. This restriction avoids the generation of short gaps in the use of

a warehouse, which are difficult to fill with another customer. To enforce these restrictions, a minimum contracting length L and a minimum waiting period for contract renewal W are defined. The binary variable y_{jt} represents whether a warehouse j is used in period t or not. New binary variables y_{jt}^s and y_{jt}^f to indicate when a contract is started and finished, respectively, are also defined. With these elements, Eq. (18)-(21) are added to the model.

$$-y_{jt} + y_{jt-1} + y_{jt}^s \geq 0 \quad \forall j, t > 1 \quad (18)$$

$$\sum_{\tau=t}^{t+L-1} y_{j\tau} \geq Ly_{jt}^s \quad \forall j, t + L - 1 \leq |T| \quad (19)$$

$$-y_{jt} + y_{jt+1} + y_{jt}^f \geq 0 \quad \forall j, t < |T| \quad (20)$$

$$\sum_{\tau=t}^{t+W} y_{j\tau} + Wy_{jt}^f \leq W \quad \forall j, t + W \leq |T| \quad (21)$$

Safety Stock with Risk-Pooling Effect

The safety stock can be expressed by Eq. (22) (Daskin et al., 2002).

$$ss = z\sigma\sqrt{L} \quad (22)$$

To represent the safety stock with risk-pooling effect using only linear constraints we need to analyze this equation. First, for a given service level and lead time, the safety stock is proportional to the absolute variance σ , which is also proportional to the demand. Additionally, to account for the risk-pooling effect, the proportionality constant must decrease with the number of customers served, which is indirectly also represented by the demanded amount. The safety stock can then be approximated by a piecewise-linear function (Figure 3). Eq (23)-(26) are the constraints to model the function.

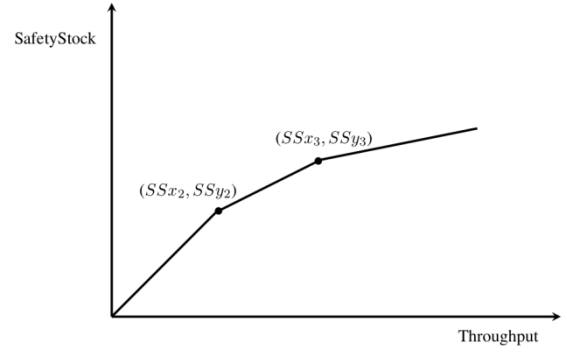


Figure 3. Safety stock piecewise-linear function

$$\sum_k x_{jkpt} = \sum_n SSx_{jktn} \lambda_{jptn} \quad \forall j, p, t \quad (23)$$

$$SS_{jpt} = \sum_n SSy_{jptn} \lambda_{jptn} \quad \forall j, p, t \quad (24)$$

$$\sum_n \lambda_{jptn} = 1 \quad \forall j, p, t \quad (25)$$

$$\lambda_{jptn} \in SOS2 \quad (26)$$

Tightening Constraints

The various features considered by the model make it more realistic but at the same time harder to solve. This is why additional effort needs to be made to solve larger instances. The first alternative explored is to include valid inequalities in the formulation that are not strictly required to obtain the optimal solution, but contribute to strengthening the relaxation, and thus, potentially solve the problem faster. Four families of tightening constraints were studied. However, only one of them resulted in a modest speed up in the solution time. Namely,

$$TCap_m u_{jktm} \leq \sum_p x_{jkpt} \quad \forall j, k, t, m \quad (27)$$

Equation (27) illustrates the valid inequalities for the warehouse-customer link, similar constraints are added for plant-warehouse and warehouse-warehouse. Constraints from Eq. (27) provide a tighter upper bound for the transportation units. They indicate that the number of transportation units used in a given link at a specific time periods will be at most the number of units that would be used if that transportation mode is unique.

Simplifying Approximations

Another strategy to decrease the solution time is to make reasonable assumptions to simplify the MILP model to obtain approximate solutions. Two formulations are

proposed based on assumptions of customer service policies.

The first simplified formulation assumes that a given customer receives a given product from a single warehouse. In the following, we will refer to this formulation as JKP, because only one of the combinations warehouse-customer-product is allowed (single-sourcing). For example, if a customer demands products A and B, it could receive product A from one warehouse (W1), and product B from another warehouse (W2), but it could not receive the same product from two separate warehouses. The assumption is reasonable because products supply tend to follow minimum plant-warehouse-customer cost routes. The deviation from this assignment only occurs when limitations of capacity are reached. On the other hand, the network design is primarily driven by transportation costs on the warehouse-customer side. The modeling effect is that the variable that represents the flow between a warehouse j and a customer k for a given product p in time period t , x_{jkpt} , is replaced by the term $D_{kpt}z_{jkp}$, product between a new binary variable, z_{jkp} , and the Demand, D_{kpt} . An additional constraint is needed to ensure the combination warehouse-customer-product is unique. The variable z_{jkp} takes a value of one if warehouse j supplies product p to customer k

$$\sum_j z_{jkp} = 1 \quad \forall k, p \quad (28)$$

It is important to observe that with the JKP formulation, a large number of continuous variables, x_{jkpt} , are replaced by smaller yet significant number of binary variables, z_{jkp} . Thus it is not straightforward to predict a decrease in solution time. However, our experiments, presented in the next section show that the impact of the reformulation is indeed positive. The second observation is that since this model represents a restriction of the original model, the objective value provides a valid upper bound cost of the original problem. This bound is typically no more than 1% higher than the optimal cost.

Taking this idea further, we can also assume that a customer receives all its demanded products from a given warehouse. In this formulation (JK), the binary variable z_{jk} indicates this assignment. As before, the variable x_{jkpt} is replaced by $D_{kpt}z_{jk}$, but additionally the transportation units in the warehouse-customer links can be precalculated offline, eliminating the integer variable u_{jktm} . In this way, the number of binary variables added is much less than before. Furthermore, a large number of continuous and integer variables is eliminated. Since the assumption is even more restrictive the resulting objective value yields an upper bound to both, the original problem and the JKP formulation.

Case Studies

To illustrate the importance of considering discrete transportation costs a case study with 8 plants, 10 warehouses, 6 customers, 5 products, 24 time periods and 4 transportation modes is presented. The problem was solved using a continuous transportation costs formulation (proportional to transported amount), and discrete transportation costs (cost per transportation unit). The objective value and solution time is presented in Table 2, whereas Figures 4 and 5 illustrate the resulting supply chain networks

Table 2. Comparison between considering discrete and continuous transportation costs

Instance	Continuous Freight Cost		Discrete Freight Cost	
	Objective	CPU(s)	Objective	CPU(s)
	T6C4P1	84.1	53	262.4

The results show a very large difference in both solution time and objective value. This indicates then, even though the continuous costs model can solve much faster than the discrete transportation costs model, it fails to correctly estimate the costs and design the optimal network. It also fails to identify the mix of transportation modes used, because if the number of units available of each mode is not restricted it will always select the lowest cost mode. For these reasons, it is very important to consider discrete transportation costs in a supply chain design model to obtain the optimal design and plan.

The second case study analyzes the effect of tightening constraints and the simplified formulations JKP and JK. They were evaluated in instances of different sizes. The results are presented in Table 3. "Orig" indicates the original formulation. "Orig-t" indicates the original formulation with the tightening constraints, and JK and JKP are the simplified formulations. As seen in Table 3, the introduction of tightening constraints yields a small reduction in solution time for instances C10P10T24 and C15P15T36. There is no reduction in solution time by using the simplified formulations for the smallest instance, C10P10T12. However, up to 95% reductions are observed for the larger problems. The objective values of the JK and JKP formulations are very close to the optimum.

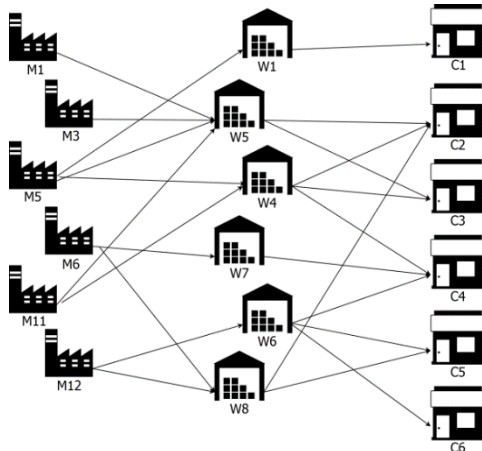


Figure 4. Optimal network for continuous cost model

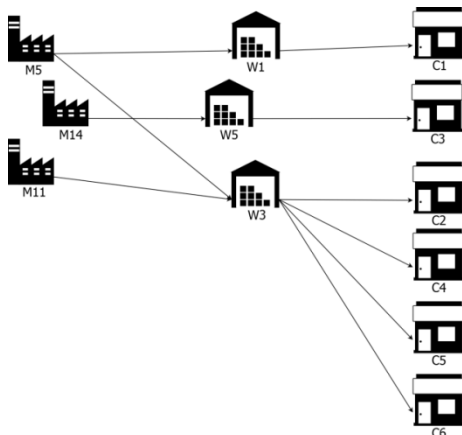


Figure 5. Optimal network for discrete cost model

Table 3. Case study 2 objective values and solution times

Instance		Orig	Orig-t	JKP	JK
C10P10T12	Obj	2816	2816	2816	2823
	CPU(s)	13	13	19	14
C10P10T24	Obj	5458	5458	5447	5453
	CPU(s)	583	579	60	97
C15P15T36	Obj	10353	10353	10353	10434
	CPU(s)	9472	9461	477	408

Conclusions

In this paper we have addressed the optimal network design for a supply chain with seasonal demand as a facility location problem. The best formulation to model the distinctive characteristics of the supply chain under study was identified among several options and solved for a mid-size case study.

The use of integer variables resulted in the most efficient formulation to address discrete transportation costs. The safety stock was modeled using a piecewise-linear approximation, and specific contracting policy constraints were derived from propositional logic.

The importance of using discrete transportation costs was illustrated with the first case study. Most real applications have this kind of cost structures, yet most

models developed simplify this by considering that the costs are proportional to the transported amount. We have showed that this simplification can result in poor supply chain network designs and incorrect costs estimations.

All the features considered contribute to have more realistic models, especially when outsourcing logistic operations. But at the same time they pose a challenge in solving the optimization model. The first steps towards solving larger problems are presented. Valid inequalities that help to tighten the relaxation were derived. It was shown that the simplified formulations JKP and JK help significantly to reduce the solution time, allowing to solve larger problems with a small increase in the objective value. Larger problems will require designing efficient decomposition algorithms.

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