Optimizing Inventory Policies in Process Networks under Uncertainty

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Abstract

We address the inventory planning problem in process networks under uncertainty through stochastic programming models. The scope of inventory planning requires the formulation of multiperiod models to represent the time-varying conditions of industrial process, but the multistage stochastic programming formulations are often too large to solve. We propose a policy-based approximation of the multistage stochastic formulation that avoids anticipativity by enforcing the same decision rule for all scenarios. The proposed formulation includes the logic modeling inventory policies, and it is used to find the parameters that offer the best expected performance. We propose policies for inventory planning in process networks with arrangements of inventories in parallel and in series. We compare the inventory planning strategies obtained from the policy-based formulation with the analogous two-stage approximation of the multistage stochastic program. Sequential implementation of both planning strategies in receding horizon simulations show the advantages of the policy-based model, despite the increase in computational complexity.

Keywords: Inventory planning, Process network, Stochastic programming, Inventory policies, Receding horizon

1. Introduction

Inventory planning is a critical aspect of enterprise-wide optimization [16]. Inventories are used in production and logistic networks to coordinate supply cycles and to mitigate the risks associated with uncertainty. The importance of inventory management in industrial applications derives from the effect of stockouts in the levels of customer satisfaction and the impact of stock in the economic balance of companies. Remarkably, the value of U.S. inventories was

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estimated to be over $1,707 billion in December 2013 [50], and the opportunity cost ascribed to capital invested in inventories added up to $434 billion in 2012 [54]. Therefore, the potential savings from stockout prevention and inventory related cost offer a huge opportunity for optimization.

Many strategies have been proposed to manage inventories since Harris [17] introduced the Economic Order Quantity (EOQ) model in 1913. The EOQ model was developed to balance ordering and holding cost for problems with a deterministic demand rate. Classical models for inventory management with uncertain demand include continuous-review (r,Q) policies and periodic-review basestock policies; the main purpose of these models is to minimize the expected cost of replenishment and stockouts, since complete satisfaction of uncertain demand might be too expensive or impossible.

One of the main advantages of the classical models is that they prescribe a policy for inventory management that is easy to implement. In fact, these policies are often optimal under assumptions satisfied by simple inventory management problems. Therefore, policies are in practice the method of choice to plan inventories in most industrial applications. However, the complexity of production networks limits the suitability of the classical models for inventory management in production processes. The main complications for inventory planning in process networks arise from the network topology, the limitations in production capacity, and the multiple sources of uncertainty.

It is common practice in industry to allocate storage units at different stages of the network in order to decouple the production of successive sections. The role of inventory is to buffer temporal mismatches among supply availability, processing rates, and demand. In addition to the raw material and final product inventories that are used to hedge against external uncertainties, production networks also include intermediate inventories that protect against the variability in processing rates. The importance of intermediate inventories resides in their ability to reduce the interdependence of processing units, to delay the formation of bottlenecks, and to increase capacity utilization.

The interest for the control of intermediate inventories in production processes can be traced back to the work by Simpson [42] in the 1950's. However, few methodologies have been proposed for inventory planning under uncertainty in continuous process networks. In this article, we focus on developing stochastic programming formulations that leverage the nature of the inventory planning problem. We propose a new approach that includes the logic of inventory policies in a mathematical programming framework with the purpose of finding optimal policy parameters. The idea is to combine the advantages of logic-based mathematical programming with the pragmatism derived from inventory management theory. This approach for inventory optimization is completely novel and offers significant benefits for production planning in complex networks. We show that using policies for inventory management in process networks has advantages over multistage or two-stage stochastic programing techniques. From the modeling perspective, policies offer an alternative way to avoid anticipativity that can be used on arbitrary sets of scenarios. From the industrial perspective, policies are attractive because they are intuitive and easy to implement.

The reminder of the article is organized as follows. In Section 2, we review the publications that are most relevant to our work. Section 3 introduces the inventory planning problem that we address. The method that we propose to solve the problem and to evaluate the solutions is outlined in Section 4. In Section 5, we present a motivating example that illustrates the particularities of the stochastic inventory planning problem. Section 6 presents the optimization model for single-echelon basestock policies. In Section 7 we revisit the motivating example to compare the inventory plans obtained from different stochastic programming models. Section 8 presents a general model for stochastic inventory planning in process networks. Sections 9 and 10 propose policies for inventory planning in process networks with inventories in parallel and in series, respectively. In Section 11, we present a simulation approach to evaluate the performance of inventory planning strategies. Sections 12 and 13 implement the proposed inventory planning models in two different examples. Finally, Section 14 presents the conclusions.

2. Literature review

Management of intermediate inventories has been addressed in the literature of multi-echelon supply chains, which was initiated with the seminal work of Clark and Scarf [4]. They proved that basestock policies are optimal for the average cost of multi-echelon serial systems with stationary stochastic demand, convex cost function, and finite horizon. Later, Federgruen and Zipkin [12] demonstrated the optimality of basestock policies in the infinite horizon. A recursive algorithm to calculate optimal basestock levels in the serial and assembly multi-echelon system with linear costs was developed by van Houtum and Zijm [52]. A simpler procedure yielding lower and upper bounds on the echelon cost functions was developed by Shang and Song [40]; they also present a heuristic for approximating optimal basestock levels that performs surprisingly well in practice.

The derivation of optimal policies for inventory management in networks with general topologies is a challenging task. The analysis of multi-echelon assembly systems presented by Rosling [36] showed that their optimal basestock policies can be obtained from an equivalent serial system. For multi-echelon distribution systems, basestock policies have only been proved to be optimal under the assumption that stockouts occur with equal probability at the downstream installations [6]. Under this *balancing assumption*, optimality of basestock policies has been proved for two-echelon systems [11, 12] and for multi-echelon systems [5].

The application of multi-echelon basestock policies to supply chain design is based on two models: guaranteed service-time and stochastic service-level. The guaranteed service-time model strategically locates safety inventories to satisfy the maximum product requirements that installations are committed to satisfy during their net lead time. The model was initially developed by Kimball [25] for a single-stage system, and implemented in serial systems by Simpson [42]. Extensions of the guaranteed service-time model for safety stock placement in assembly and distribution networks have been developed for bounded demands [15, 14] and for normally distributed demands [19, 20]. The alternative stochastic service-level model developed by Lee and Billington [29] locates inventories to offer prescribed service levels at the installations of a decentralized supply chain; basestock levels are obtained from the characterization of random delays experienced by installations as a result of shortages in the upstream stages.

Ettl et al. [8] developed expressions for the actual lead time in a multi-echelon supply chain by approximating the dynamics of inventory levels with queuing models; they also included their inventory model in an optimization framework to minimize the total inventory cost. The use of queuing models to characterize production and distribution networks started with the work of Jackson [21]. The advantage of queuing networks is that they allow modeling the dynamics of inventories in networks with finite processing capacity. The most influential queuing models of manufacturing systems characterize them with product-form solutions that can be found for a restrictive class of networks, from which Jackson networks are representative. An exceptional model capturing the dynamics of basestock policies in serial servers was developed by Lee and Zipkin [30]; they showed that the serial system can be described exactly for some special cases and they developed approximations for the general case.

The characterization of optimal policies for capacitated production-inventory systems with stationary demand was presented by Federgruen and Zipkin [9, 10]. They showed that under the usual assumptions, a modified basestock policy is optimal in the infinite horizon for the average and discounted cost criteria, and also for the discounted cost criterion in a finite horizon. The modification of the classic basestock policy accounts for the capacity limitation by truncating the replenishment when the order-up-to quantity cannot be fulfilled. An algorithm to calculate optimal basestock levels and the corresponding costs for capacitated multi-echelon systems in the infinite horizon was developed by Tayur [47] using a sequence of uncapacitated models that converge to the capacitated system. A more general simulation-based method to find optimal basestock levels in capacitated multi-echelon systems was presented by Glasserman and Tayur [13]; their *Infinitesimal Perturbation Analysis* (IPA) estimates the sensitivity of the cost function with respect to the policy parameters and use them to recursively improve the basestock levels.

Most of the literature about inventory management in chemical process networks is related to deterministic systems. Karimi and Reklaitis [24] recognized the importance of intermediate storage for batch and semicontinuous processes, and derived expressions to find optimal storage capacities according to the periodicity of the production processes. Other models for multiproduct batch plants have included uncertainty in the design problem [53, 39, 18], but they have not considered inventory management in their formulations. The integration of batch plant design and scheduling was addressed by Subrahmanyam et al. [44] using a decomposition approach that iterates between a design superproblem and scheduling subproblems. Petkov and Maranas [33] addressed the optimal design and operation of batch plants with normally distributed demand for multiple products assuming a single-product campaign production mode; they exploited the properties of normal distributions to find the optimal operating policy corresponding to the potential designs.

Multi-echelon policies have also been applied for inventory management in the process industry. Jung et al. [22] developed a simulation-optimization approach in which safety stock levels are determined in a linear program and evaluated using discrete-event simulation. The proposed approach can accommodate diverse network structures and uncertainty characterizations. Recently, Chu et al. [2, 3] presented a similar approach that uses agent-based simulations to generate linear inequalities that are added to the LP planning problem to enforce the service level constraint; this approach has been used for reactive scheduling and multi-echelon inventory planning.

The guaranteed service-time model was implemented by You and Grossmann [56] for the design of chemical supply chains with uncertain demand; they extended the guaranteed service-time methodology for production planning and inventory management in dedicated chemical networks that include capacity constraints [57]. In a subsequent publication, dedicated and flexible processes are considered simultaneously by including a cyclic scheduling model that determines the sequence and duration of the flexible processes [58]. An MILP formulation for the optimal design of chemical networks with uncertainty in supply, demand, and random failures was developed by Terrazas-Moreno et al. [48, 49]. Their analysis considers the impact of slack production capacity and the effect of intermediate inventories in the reliability of the production network. The formulation proposed by Terrazas-Moreno et al. [48] allows including diverse characterizations of uncertainty as exemplified by their description of random failures using a Markov process.

An alternative approach for inventory management in production and distribution networks has leveraged control theory for sequential decision-making. Bose and Pekny [1] proposed using Model Predictive Control (MPC) for planning and scheduling of supply chain activities; their framework included forecasting, optimization, and simulation modules. Perea-López et al. [31, 32] modeled the dynamics of supply chains by considering flows of material and information. In a first article [31], they implemented site-dependent control laws to simulate the behavior of decentralized supply chains in closed loop. In a second article [32], they developed a discrete-time model of the supply chain dynamics and used MPC to plan production and distribution in a rolling horizon. The integration of scheduling and control for coordination of production and distribution has been recently addressed by Subramanian et al. [46]; their model characterizes the state of the system according to inventory levels and compare three MPC approaches that manipulate orders and shipments. In a related article, Subramanian et al. [45] proposed a state-space model for scheduling that describes the system with the levels of inventory, the tasks in progress, and their starting time; shipments, yield variations, delays, and unit breakdown are considered disturbances in the model.

Another body of literature related to our research advocates for the use of stochastic programming in supply chain design and operation. Tsiakis et al. [51] proposed a MILP formulation for the design of multi-echelon supply chains considering scenarios with uncertain demand. You et al. [55] developed a two-stage stochastic programming model for supply chain planning under uncertainty with risk management. Jung et al. [23] proposed a multistage stochastic programming formulation for multi-period supply chain planning; their solution method iterates between a rolling horizon simulation and an outer loop that improves the safety stock targets using a gradient-based search.

Stochastic programming problems with a very large number of scenarios have been successfully solved through Sample Average Approximation (SAA) [41, 26]. SAA is a framework to approximate the optimal expected value of a stochastic program based on the solution of smaller problems with randomly sampled scenarios; the method provides statistical bounds on the expectation of the optimal objective value. Santoso et al. [37] implemented SAA for the optimal design of a supply chain with uncertain supply, capacity, cost structure, and demand. The minimum-cost design of a supply chain with a complex topology was formulated as a two-stage stochastic program by Schütz et al. [38]; in their formulation, the design is decided in the first stage and the operation is modeled in the second stage. An implementation of SAA for the design of resilient supply chains was presented by Klibi and Martel [27]; their stochastic programming formulation considers disruptions and other types of uncertainty in the scenarios.

3. Problem description

Inventory management involves decisions related to the replenishment and depletion of inventories. In continuous process networks, inventory decisions are closely related to production planning because most units are simultaneously internal suppliers and consumers. The complexity of chemical production networks requires storage of raw materials to guard against supply variability, intermediates to avoid the formation of bottlenecks, and final products to hedge against demand uncertainty. The role of intermediate inventories is widely understood in industrial applications but few methodologies have been proposed to optimize their management strategies in continuous process networks with complex topologies and capacity constraints.

This work addresses the inventory planning problem in continuous process networks with uncertainty in supply, available production capacity, and demand. We impose no restrictions on the characterization of the uncertain parameters other than the availability of discrete-time forecasts. Then, given a process network with known structure, our goal is to propose planning strategies that minimize the expected costs of inventory holding and stockouts in a finite horizon.

4. Outline of solution and result evaluation methods

The inventory planning problem under uncertainty can be formulated as a stochastic programming (SP) problem where production and inventory decisions are optimized to obtain the plan with minimum expected cost. In multiperiod problems with a discrete number of scenarios, the optimal solution of such a problem can be obtained by solving a multistage SP formulation. However, because of the computational difficulty to solve large-scale multistage SP models, it is often necessary to approximate them with two-stage SP formulations. Two-stage SP models are significantly easier to solve, but they do not capture the sequence in which information about uncertain parameters is revealed, which might deteriorate the quality of their solutions. We propose an alternative approximation of the multistage SP model that avoids anticipating the outcomes of uncertainty by enforcing inventory policies for all scenarios.

We develop a logic-based SP formulation that integrates inventory policies in a mathematical programming framework. In order to optimize these policies, we first postulate a parametric model mapping the levels of inventory in the network to replenishment and depletion actions. This parametric model is based on the logic of basestock policies and includes additional rules according to the topology of the process network. The logic-based SP formulation optimizes the parameters of the inventory policy with the objective of minimizing the expected cost over the scenarios.

Each scenario describes the trajectory of all uncertain parameter throughout the planning horizon. The scenarios can be generated by reproducing all possible trajectories in problems with discrete uncertain parameters, by simulating sample-paths from stochastic processes, from historical data, or from any other forecasting method. The probability associated to scenarios depends on the method used to generate them.

The most rigorous evaluation of the quality of a stochastic solution requires comparing the expected cost obtained by implementing it with the optimal expected cost of the multistage SP model. This is the approach that we follow for the motivating example in Sections 5 and 7. The alternative for problems with too many scenarios is to compare different decision strategies using closed-loop Monte Carlo simulations. These simulations involve a sequential decision-making process that implements the first-period decisions recursively. The simulation horizon specifies the number of times that decisions are made and implemented. Closed-loop Monte Carlos simulations yield a cost associated with the decision-making strategy, but this cost is a random outcome. Therefore, several replications are required to estimate the expected simulation cost and to compare the quality of different decision-making strategies. We use this approach to evaluate the inventory planning strategies presented in Sections 9 and 10.

5. A motivating example

We present an small motivating example to illustrate the proposed inventory planning approach. The problem considers production planning and inventory management in a production-inventory system with uncertain demand. The system includes a single processing unit with deterministic production capacity, a storage unit with unlimited capacity, and stochastic demand. The planning problem has a discrete time horizon with 11 periods spanning from t_0 to t_{10}.

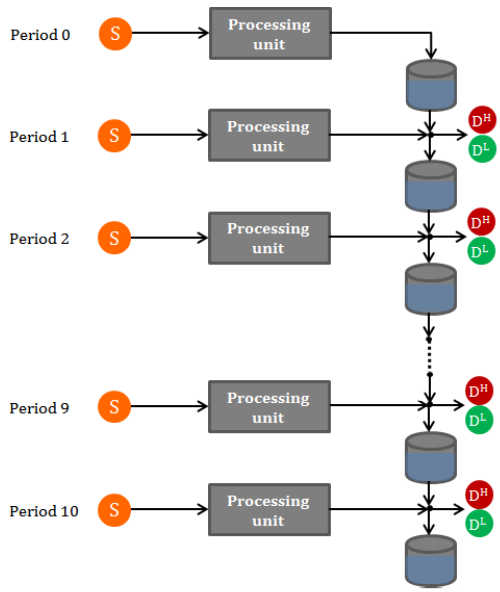


Figure 1: Schematic representation of the motivating example.

Demands are independent and identically distributed (iid) uncertain parameters characterized by a discrete uniform probability distribution in periods t_1 to t_{10}. A schematic representation of the motivating example is presented in Fig. 1.

We consider the case study in which supply (S) is unlimited, available production capacity (C) is 100 units of product per period, and demand can be either 110 or 90 units of product per period ($D^H = 110$, $D^L = 90$). It is worth noticing that demand can be fully satisfied by accumulating inventory in the initial time period (t_0), even if the outcome of uncertain demand is high in periods t_1 to t_{10}.

The objective of the planning problem is to minimize the expected costs associated to inventory holding and stockouts. Stockouts are calculated according to the backorders model that carries out unsatisfied demands to the next time period. We use a unit holding cost (H) of \$5/unit-period and a unit backorder cost (P) of \$15/unit-period.

The planning problem entails a sequential decision-making process in which new information becomes available as uncertainty is revealed with time. The problem has a discrete representation of time and a finite support for the uncertainty space; therefore, we can formulate it as a multistage SP problem. The multistage SP model is presented in Eqns. (1)-(6).

$$\min \quad Hx_{t_0} + \mathop{\mathbb{E}}_{\xi \in \Xi} \left[\sum_{t \in T \setminus \{t_0\}} Hx_{\xi,t} + Pb_{\xi,t} \right] \tag{1}$$

$$\text{s.t.} \quad q_{\xi,t} + u_{\xi,t} = C \qquad \forall\, t \in T,\ \xi \in \Xi \tag{2}$$

$$[x_{\xi,t} - b_{\xi,t}] = [x_{\xi,t-1} - b_{\xi,t-1}] + q_{\xi,t} - D_{\xi,t} \qquad \forall\ t \in T,\ \xi \in \Xi \quad (3)$$

$$x_{\xi,t} = x_{\xi',t}, \quad b_{\xi,t} = b_{\xi',t}, \quad q_{\xi,t} = q_{\xi',t}, \quad u_{\xi,t} = u_{\xi',t}$$
$$\forall\ t = \{t_0\},\ (\xi, \xi') \in \Xi \times \Xi \quad (4)$$

$$x_{\xi,t} = x_{\xi',t}, \quad b_{\xi,t} = b_{\xi',t}, \quad q_{\xi,t} = q_{\xi',t}, \quad u_{\xi,t} = u_{\xi',t}$$
$$\forall\ t \in T \setminus \{t_0\},\ (\xi, \xi') \in \Gamma_t \quad (5)$$

$$x_{\xi,t},\ b_{\xi,t},\ q_{\xi,t}, u_{\xi,t} \in \mathbb{R}^+ \qquad \forall\ t \in T,\ \xi \in \Xi \quad (6)$$

where T is the set of time periods (t), Ξ is the set of scenarios (ξ), and Γ_t is the set of scenario pairs with the same outcomes of the uncertain parameters up to time t. This set is used to enforce that decisions can only depend on the outcomes of past stages, which is the non-anticipativity condition. The formal definition of Γ_t for the example is given by Eqn. (7).

$$\Gamma_t := \left\{(\xi, \xi') : \quad (\xi, \xi') \in \Xi \times \Xi, \quad (D_{\xi,t_1}, D_{\xi,t_2}, ..., D_{\xi,t}) = (D_{\xi',t_1}, D_{\xi',t_2}, ..., D_{\xi',t})\right\} \quad (7)$$

The multistage SP formulation given by Eqns. (1)-(6) is known as the *explicit representation* because it includes copied variables for each scenario and Non-Anticipativity Constraints (NAC) relating them. The model denotes end-of-period inventory level with variables $x_{\xi,t}$ and end-of-period stockouts with variables $b_{\xi,t}$; processing rate is denoted with variables $q_{\xi,t}$ and underutilization with $u_{\xi,t}$. The objective function is given by Eqn. (1). In the first period, t_0, only holding cost is considered because there is no demand; in subsequent periods, holding and backorder costs are incurred. Eqn. (2) represents the capacity constraint of the processing unit, where the slack variable $u_{\xi,t}$ denotes underutilization. The mass balance in the storage unit is modeled with Eqn. (3); production in a period is considered instantaneous. Non-anticipativity of the decisions is enforced with Eqns. (5)-(4). The domains of the variables are presented in Eqn. (6).

The multistage SP formulation for this motivating example describes in scenarios the possible trajectories of demand; there are 1,024 scenarios corresponding to the sequences of demand from period t_1 to t_{10}. Despite the large number of scenarios, this multistage SP model is a Linear Program (LP) that can be solved with any commercial solver. The optimal solution specifies the value of 45,056 variables in the explicit representation or 8,188 variables in an implicit formulation without copied variables. However, the same optimal solution can be described in much simpler form using a basestock policy.

The capacitated single-echelon basestock policy establishes rules to operate the system according to the inventory level. The basestock level indicates the ideal level of inventory in a given period. Following the basestock policy, production capacity and inventory are first used to satisfy demand; surplus capacity is used to intend replenishing inventory up to the basestock, but no inventory in excess of the basestock level is hold. The optimal basestocks for the motivating example and the corresponding expected costs are presented in Table 1.

Period	Basestock	Expected costs [$]		
		Holding	Backorder	Total
t_0	30	150.00	0.00	150.00
t_1	20	100.00	0.00	100.00
t_2	20	75.00	0.00	75.00
t_3	20	62.50	0.00	62.50
t_4	20	56.25	18.75	75.00
t_5	20	50.00	28.13	78.13
t_6	20	46.88	46.88	93.76
t_7	10	27.34	58.59	85.93
t_8	10	19.53	76.17	95.70
t_9	10	17.77	100.19	117.98
t_{10}	0	0.00	121.00	121.00
Total:		605.27	449.71	1,054.98

Table 1: Optimal basestock levels and costs for the motivating example.

6. Capacitated single-echelon basestock policy

It is not always easy to infer the optimal basestock levels from the solution of the multistage SP formulation. Nevertheless, the simplicity and intuitive appeal of inventory policies advocates for a general framework to obtain optimal basestock levels. Let us denote by y_t the basestock level of the single-echelon system at time t. Then, the sequence of events involved in implementation of the basestock policy can be described as follows:

1. Random demand ($D_{\xi,t}$) is realized.
2. Production capacity (C) and carried over inventory ($x_{\xi,t-1}$) are used to satisfy demand ($D_{\xi,t}$) and backorders ($b_{\xi,t-1}$).
3. Surplus capacity is used to replenish inventory up to the basestock level (y_t).
4. Inventory level ($x_{\xi,t}$) and backorders ($b_{\xi,t}$) are updated.
5. Holding or stockout cost is calculated.

The logic describing the operation of the basestock policy in a capacitated single-echelon systems is simple. It can be characterized with the conditions given by Eqns. (8)-(9).

- Backorders ($b_{\xi,t}$) are allowed if there is no inventory:

$$b_{\xi,t} = \begin{cases} 0, & \text{if } x_{\xi,t} > 0 \\ D_{\xi,t} + b_{\xi,t-1} - x_{\xi,t-1} - C_t, & \text{if } x_{\xi,t} = 0 \end{cases} \tag{8}$$

- Underutilization ($u_{\xi,t}$) is allowed if inventory is at basestock level:

$$u_{\xi,t} = \begin{cases} 0, & \text{if } x_{\xi,t} < y_t \\ x_{\xi,t-1} + C - D_{\xi,t} - b_{\xi,t-1}, & \text{if } x_{\xi,t} = y_t \end{cases} \tag{9}$$

In order to include the basestock policy in a mathematical programming formulation, we divide the state-space of the system in three discrete states: empty inventory, intermediate level, and full inventory. In each state, the logic dictates the processing rate and inventory management plan according to a different rule. This logic can be modeled with the disjunctions presented in Eqn. (10),

$$\begin{bmatrix} x_{\xi,t} = 0 \\ b_{\xi,t} \geq 0 \\ u_{\xi,t} = 0 \end{bmatrix} \vee \begin{bmatrix} 0 < x_{\xi,t} < y_t \\ b_{\xi,t} = 0 \\ u_{\xi,t} = 0 \end{bmatrix} \vee \begin{bmatrix} x_{\xi,t} = y_t \\ b_{\xi,t} = 0 \\ u_{\xi,t} \geq 0 \end{bmatrix} \qquad \forall \;\; t \in T, \; \xi \in \Xi \tag{10}$$

where the term on the left models the basestock policy with an empty inventory, the term on the center with an intermediate level, and the term on the right with a full inventory. Strict inequalities modeling intermediate levels ($0 < x_{\xi,t} < y_t$) can be implemented in the mathematical programming environment with epsilon precision ($\epsilon \leq x_{\xi,t} \leq y_t - \epsilon$).

The formulation enforcing a basestock policy for inventory management in the motivating example is obtained by replacing NAC constraints (5) with the logic presented in Eqn. (10). The most obvious advantage of this logic-based SP formulation is that its solution can be easily characterized with the basestock levels (y_t). The formulation is a Generalized Disjunctive Program (GDP) that can be seen as a multiperiod SP formulation with piece-wise linear decision rules for inventory management [28].

The GDP model can be reformulated as a Mixed-Integer Linear Program (MILP) by introducing binary variables; for notational convenience, we denote binary variables with a hat ($\hat{}$) throughout the article. Binary variables $\hat{\mathrm{x}}^0_{\xi,t}$ and $\hat{\mathrm{x}}^y_{\xi,t}$ indicate if the inventory is empty or at the basestock level, respectively. The conditions defining these variables are presented in Eqns. (11)-(14),

$$x_{\xi,t} \leq M\left(1 - \hat{\mathrm{x}}^0_{\xi,t}\right) \qquad \forall \; \xi \in \Xi, \; t \in T \tag{11}$$
$$x_{\xi,t} \leq y_t \qquad \forall \; \xi \in \Xi, \; t \in T \tag{12}$$
$$x_{\xi,t} \geq y_t - M\left(1 - \hat{\mathrm{x}}^y_{\xi,t}\right) \qquad \forall \; \xi \in \Xi, \; t \in T \tag{13}$$
$$\hat{\mathrm{x}}^0_{\xi,t} + \hat{\mathrm{x}}^y_{\xi,t} \leq 1 \qquad \forall \; \xi \in \Xi, \; t \in T \tag{14}$$

where the parameter M is an upper bound on the basestock level, Eqn. (11) forces the inventory to be empty if variable $\hat{\mathrm{x}}^0_{\xi,t}$ equals one, Eqns. (12)-(13) forces the inventory to be at basestock level if $\hat{\mathrm{x}}^0_{\xi,t}$ equals one, and Eqn. (14) allows selecting only one of these states per scenario and time period.

The logic presented in Eqn. (10) is completed with Eqns. (15)-(16),

$$b_{\xi,t} \leq M\hat{\mathrm{x}}^{0}_{\xi,t} \qquad \forall\, \xi \in \Xi,\ t \in T \tag{15}$$

$$u_{\xi,t} \leq M\hat{\mathrm{x}}^{y}_{\xi,t} \qquad \forall\, \xi \in \Xi,\ t \in T \tag{16}$$

where Eqn. (15) allows stockouts only when the inventory is empty, and Eqn. (16) allows underutilization only when the inventory is at basestock level.

The MILP reformulation of the logic-based SP model is obtained by replacing the NAC constraints (5) in the multistage SP model with Eqns. (11)-(16). The resulting model can be solved using any available MILP solver.

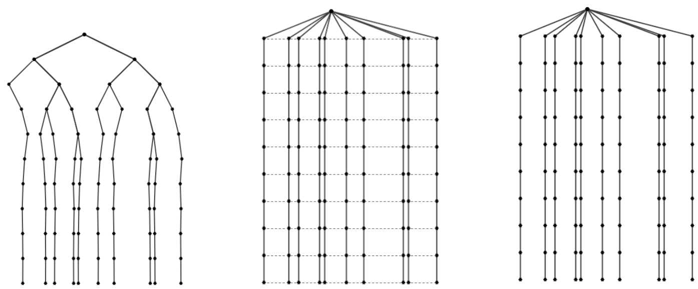
7. Motivating example revisited

Despite the convenience of establishing production and inventory management plans according to a policy, solving the logic-based SP formulation can be significantly harder than solving an LP model. Additionally, there is no guarantee that the optimal policy obtained from the logic-based SP formulation yields an expected value as good as the optimal multistage SP solution. However, large-scale multistage SP problems are also difficult to solve and often the multistage model is only an approximation of the real problem. The most common approximation is to restrict the number of scenarios in problems with a large number of discrete uncertain parameters or in problems with continuous support.

In order to asses the quality of the solutions obtained from different approximations of multistage stochastic programs, we propose a new performance metric called the *Residual Expected Value* (REV). The REV of a solution is the optimal expected value of the multistage SP problem after fixing the first-stage variables. The REV generalizes the multistage *Value of the Stochastic Solution* (VSS) to allow comparing the quality of different decision-making strategies, since VSS only compares the SP solution with the solution of the expected value problem [7].

We evaluate the performance of a decision-making strategy by measuring how much the REV deviates from the expected value obtained from the exact multistage SP formulation. Our analysis considers three decision-making strategies for the production and inventory planning problem presented in Section 5. All formulations approximate the multistage SP solution based on a model with a reduced number of sampled scenarios. In addition to the multistage SP and the logic-based SP problem, we include in our analysis the results from the two-stage SP problem. The two-stage SP problem is obtained by relaxing NAC constraints (5) of the multistage SP problem in all stages after the first. Instances of the scenario trees generated using the sampling technique are presented in Fig. 2.

The trees presented in Fig. 2 are generated by sampling 10 scenarios randomly. The multistage structure in Fig. 2a can only be recognized in the first



(a) Sampled multistage tree

(b) Sampled policy-constraint tree

(c) Sampled two-stage tree

Figure 2: Scenario trees for the motivating example

few periods. After period 5, the sampled multistage tree does not have indistinguishable scenarios, which makes it identical to the two-stage tree in Fig. 2c. On the other hand, the policy-constraint tree maintains non-anticipativity by implementing a single decision logic for all scenarios.

We compare the REV for the three SP models using different sample sizes. Each point presented in Fig. 3 was estimated with 200 sample trees generated using Latin-Hypercubes Sampling; the same 200 sample trees were used to evaluate all SP models. Fig. 3 shows that a relatively low number of scenarios is needed to obtain a good first-stage solution with the multistage and logic-based SP formulations; with 100 sample scenarios, both formulations produce a REV that is within 1% of the expected value of the full multistage SP model. The two-stage SP formulation on the other hand, does not seem to provide better solutions even with a larger number of scenarios; furthermore, the error bars indicate a high variability in its results. One of the most interesting observations from Fig. 3 is that the logic-based SP formulation outperforms the multistage SP formulation when small sample sizes are used. This might be specially relevant for stochastic programming problems with a large number of scenarios or for stochastic problems with continuous random parameters.

8. Mathematical model for stochastic inventory planning in process networks

Our model to formulate the inventory planning problem considers process networks of general topology. The transformation of raw materials into final products is achieved with a sequence of steps that are carried out in specific processing units. We denote the set of materials by M and the set of processing units by I. Three sets of parameters are considered uncertain in the formulation: available supply, available production capacity, and demand. For ease of notation, we use capitalized letters for parameters and sets, and lower-case letters for variables and indexes; all variables in this section are defined in the

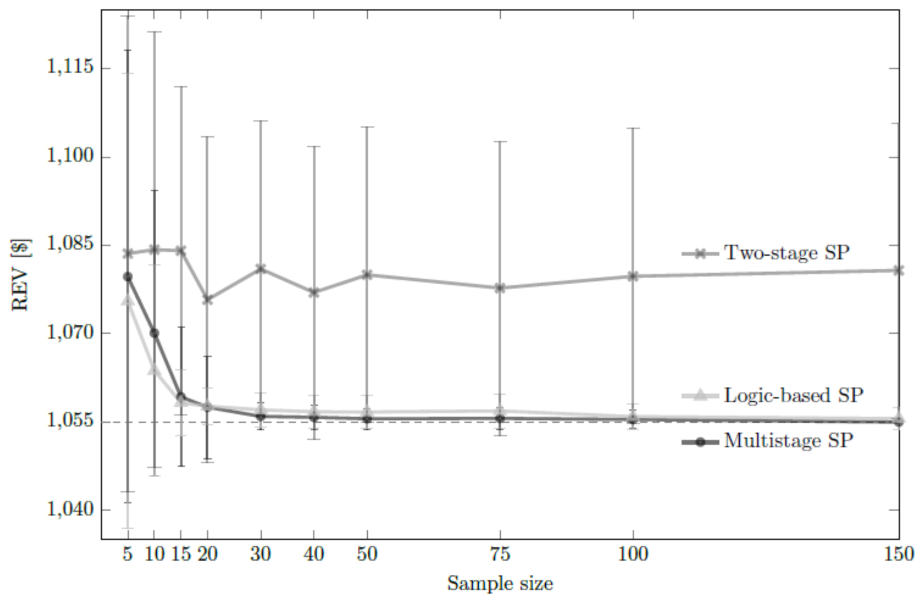


Figure 3: Residual expected value as a function of the sample size

positive real domain. The equations describing the mathematical model are presented in the remainder of this section.

8.1. Supply balances

The availability of supply is modeled with Eqn. (17). The subset of materials that are externally supplied is denoted by M^S. The amount of material m that is available as external supply at time t and scenario ξ is given by parameter $S_{\xi,t,m}$. I_m^S is the subset of processing units that receive external supply of material m. The flow of supply consumed in unit i is denoted by $f^S_{\xi,t,i,m}$, the flow that is stored as inventory by $r^S_{\xi,t,m}$, and the underutilization of supply by $v_{\xi,t,m}$.

$$S_{\xi,t,m} = \sum_{i \in I_m^S} f^S_{\xi,t,i,m} + r^S_{\xi,t,m} + v_{\xi,t,m} \qquad \forall\, \xi \in \Xi,\ t \in T,\ m \in M^S \tag{17}$$

8.2. Production capacity

The capacity of processing units is modeled with Eqn. (18). We define the available production capacity ($C_{\xi,t,i}$) as an uncertain parameter to model random variations impacting the potential throughput of processing units; the maximum capacity of a unit is always greater than its available production capacity. The processing rate is denoted by $q_{\xi,t,i}$ and the underutilization by $u_{\xi,t,i}$.

$$C_{\xi,t,i} = q_{\xi,t,i} + u_{\xi,t,i} \qquad \forall\, \xi \in \Xi,\ t \in T,\ i \in I \tag{18}$$

8.3. Consumption balance

The consumption of material m in processing unit i is modeled with Eqn. (19). The subset of materials that are consumed in unit i is denoted by M_i^{in}; the mass balance coefficient indicating the amount of material m that is consumed per unit production rate is give by parameter $A_{i,m}$. The subset of processing units feeding material m to unit i is denoted by $I_{i,m}^{up}$. The flow of material m from unit i' to unit i is $f_{\xi,t,i',i,m}$, and the amount of inventory depleted to feed unit i is modeled with variable $d_{\xi,t,i,m}$.

$$A_{i,m} q_{\xi,t,i} = f_{\xi,t,i,m}^{S} + \sum_{i' \in I_{i,m}^{up}} f_{\xi,t,i',i,m} + d_{\xi,t,i,m} \quad \forall\, \xi \in \Xi,\ t \in T,\ i \in I,\ m \in M_i^{in} \tag{19}$$

8.4. Production balance

The production of material m in processing unit i is modeled with Eqn. (20). The subset of materials that are produced in unit i is denoted by M_i^{out}; the mass balance coefficient indicating the amount of material i that is produced per unit production rate is given by parameter $B_{i,m}$. The subset of processing units receiving material m from unit i is denoted by $I_{i,m}^{down}$. The amount of inventory replenished by unit i is $r_{\xi,t,i,m}$, and the production flow that is used to satisfy demand is $f_{\xi,t,i,m}^{D}$.

$$B_{i,m} q_{\xi,t,i} = \sum_{i' \in I_{i,m}^{down}} f_{\xi,t,i,i',m} + r_{\xi,t,i,m} + f_{\xi,t,i,m}^{D} \quad \forall\, \xi \in \Xi,\ t \in T,\ i \in I,\ m \in M_i^{out} \tag{20}$$

8.5. Inventory balance

The inventory of material m is modeled with Eqn. (21). The subset of materials that can be stored is denoted by M^x. The balance includes the inventory carried-over from the last period ($x_{\xi,t-1,m}$), the replenishment from supply ($r_{\xi,t,m}^{S}$), the replenishment from processing units ($r_{\xi,t,i,m}$), the inventory used to feed processing units ($d_{\xi,t,i,m}$), and the inventory used to satisfy external demand ($d_{\xi,t,m}^{D}$). The set of units allowed to replenish the inventory of material m is denoted by I_m^r, and the set of units that can deplete inventory of material m is denoted by I_m^d.

$$x_{\xi,t,m} = x_{\xi,t-1,m} + r^S_{\xi,t,m} + \sum_{i \in I^r_m} r_{\xi,t,i,m} - \sum_{i \in I^d_m} d_{\xi,t,i,m} - d^D_{\xi,t,i,m}$$
$$\forall\ \xi \in \Xi,\ t \in T,\ m \in M^x \qquad (21)$$

8.6. Demand balance

Demand satisfaction is modeled with Eqn. (22). The subset of materials with external demand is denoted by M^D. Demand ($D_{\xi,t,m}$) and carried-over backorders ($b_{\xi,t-1,m}$) are equal to the production flow that is used satisfy demand ($f^D_{\xi,t,i,m}$), the inventory that is depleted to satisfy demand ($d^D_{\xi,t,m}$), and the end-of-period backorders ($b_{\xi,t,m}$).

$$D_{\xi,t,m} + b_{\xi,t-1,m} = \sum_{i \in I^D_m} f^D_{\xi,t,i,m} + d^D_{\xi,t,m} + b_{\xi,t,m} \quad \forall\ \xi \in \Xi,\ t \in T,\ m \in M^D \qquad (22)$$

8.7. Objective function

Different objective functions can be used in the inventory planning problem. In our formulation, we minimize the sum of expected holding and stockout costs as presented in Eqn. (23). The probability of scenario ξ is denoted by $\mathbb{P}_\xi$. The holding cost of material m at period t is denoted by $H_{t,m}$, and the penalty per unit backorder of material m at period t is denoted by $P_{t,m}$.

$$\min \quad \sum_{\xi \in \Xi} \mathbb{P}_\xi \sum_{t \in T} \left(\sum_{m \in M^x} H_{t,m} x_{\xi,t,m} + \sum_{m \in M^D} + P_{t,m} b_{\xi,t,m} \right) \qquad (23)$$

9. Policy for inventories in parallel

We propose a priority-based policy for storable materials that compete for the same replenishment resources. The basic condition is that policy parameters must be the same across scenarios. The goal of the model is to establish the optimal priorities ($\hat{z}_{n,l,m}$) and basestock levels ($y_{t,m}$) for inventories in a parallel arrangement. An illustration of a parallel arrangement with three storable materials is presented in Fig. 4.

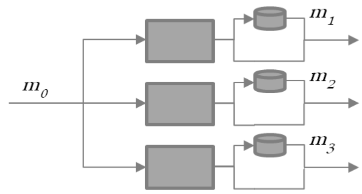


Figure 4: Parallel arrangement with m_0 as a shared resource for the replenishment of inventories m_1, m_2, and m_3.

9.1. Logic-based formulation

We denote by N the set of parallel arrangements in the process network, by $\tilde{M}_n$ the subset of storable materials that belong to parallel arrangement n, and by $R_n \subset M$ the materials that are considered shared resources for the production of $m \in \tilde{M}_n$. The set of priority levels in parallel arrangement n is L_n. The number of priorities and the number of storable materials in a parallel arrangement are set equal ($|\tilde{M}_n| = |L_n|$) with the purpose of assigning unique priority levels.

The binary variables indicating the ordering of priorities for the storable materials in a parallel arrangement are defined according to Eqn. (24).

$$\hat{\mathrm{z}}_{n,l,m} = \begin{cases} 1, & \text{if material } m \text{ has priority level } l \text{ in parallel arrengment } n \\ 0, & \text{otherwise} \end{cases} \tag{24}$$

In order to ensure that each storable material in a parallel arrangement is assigned a unique priority level, we use the exclusive -or- conditions presented in Eqn. (25)-(26),

$$\underline{\bigvee_{m \in \tilde{M}_n}} \; [\hat{\mathrm{z}}_{n,l,m} = 1] \qquad \forall\, n \in N,\; l \in L_n \tag{25}$$

$$\underline{\bigvee_{l \in L_n}} \; [\hat{\mathrm{z}}_{n,l,m} = 1] \qquad \forall\, n \in N,\; m \in \tilde{M}_n \tag{26}$$

where we express the disjunctions in terms of binary variables for notational convenience. The boolean logic can be obtained by establishing the following correspondence between binary ($\hat{\mathrm{z}}_{n,l,m}$) and boolean ($Z_{n,l,m}$) variables:
$\hat{\mathrm{z}}_{n,l,m} = 1 \;\Leftrightarrow\; Z_{n,l,m} = \text{true}$
$\hat{\mathrm{z}}_{n,l,m} = 0 \;\Leftrightarrow\; Z_{n,l,m} = \text{false}$

The priorities established by variables $\hat{\mathrm{z}}_{n,l,m}$ specify the order in which inventories in the arrangement are replenished. In particular, the material that is assigned priority $l+1$ can only be replenished if the replenishment of material with priority l is complete. Binary variable $\hat{\mathrm{w}}_{\xi,t,n,l}$ indicates that the replenishment of material with priority level l is complete in a given scenario (ξ) and time period (t). The definition of variable $\hat{\mathrm{w}}_{\xi,t,n,l}$ is given by Eqn. (27).

$$\hat{\mathrm{w}}_{\xi,t,n,l} = \begin{cases} 1, & \text{if replenishment of material with priority } l \text{ is complete} \\ 0, & \text{otherwise} \end{cases} \tag{27}$$

The completion of replenishment for material with priority level l implies that no additional upstream materials shared in the parallel arrangement are needed to replenish this inventory. If we denote by I_m^r the set of units that can replenish the inventory of material m, variable $\hat{\mathrm{w}}_{\xi,t,n,l}$ must satisfy the condition given by Eqn. (28).

$$\left[\begin{array}{c} \hat{\mathrm{z}}_{n,l,m} = 1 \\ x_{\xi,t,m} < y_{t,m} \\ \bigvee_{i \in I_m^r} \left[\begin{array}{l} u_{\xi,t,i} > 0 \\ \hat{\mathrm{g}}_{\xi,t,i,m} = 0 \quad \forall\, m \in M_i^{in} \setminus \{R_n\} \end{array} \right] \end{array} \right]$$
$$\implies \quad \hat{\mathrm{w}}_{\xi,t,n,l} = 0 \qquad \forall\, \xi \in \Xi,\ t \in T,\ n \in N,\ l \in L_n,\ m \in \tilde{M}_n \tag{28}$$

The implication presented in Eqn. (28) states that the replenishment of inventory with priority level l cannot be considered complete if the inventory level ($x_{\xi,t,m}$) is below the basestock ($y_{t,m}$), and there is available capacity ($u_{\xi,t,i} > 0$) and upstream materials ($\hat{\mathrm{g}}_{\xi,t,i,m} = 0$) for the units ($i \in I_m^r$) that can replenish it; we exclude the shared resource (R_n) from the set of upstream materials (M_i^{in}) required for the replenishment because their shortage does not relax the implication. Binary variables $\hat{\mathrm{g}}_{\xi,t,i,m}$ indicate if there is an upstream shortage of material m that does not allow increasing the processing rate in unit i. The logic establishing material shortage is given by Eqn. (29).

$$[x_{\xi,t,m} > 0] \ \vee\ [v_{\xi,t,m} > 0] \bigvee_{i' \in I_{i,m}^{up}} \left[\begin{array}{l} u_{\xi,t,i'} > 0 \\ \hat{\mathrm{g}}_{\xi,t,i',m} = 0 \quad \forall\, m \in M_{i'}^{in} \end{array} \right]$$
$$\implies \hat{\mathrm{g}}_{\xi,t,i,m} = 0 \qquad \forall\ \xi \in \Xi,\ t \in T,\ i \in I_m^{cons},\ m \in M \tag{29}$$

Expression (29) does not allow indicating shortage of material m for the units that consume it ($i \in I_m^{cons}$), if there is available inventory, supply underutilization, or the upstream units capable of producing it ($i' \in I_{i,m}^{up}$) are not fully utilized nor in shortage of the materials they consume (M_i^{in}).

The priorities for the replenishment are enforced with Eqns. (30)-(31).

$$\hat{\mathrm{w}}_{\xi,t,n,l} = 0 \implies \hat{\mathrm{w}}_{\xi,t,n,l+1} = 0 \qquad \forall\, \xi \in \Xi,\ t \in T,\ n \in N,\ l \in L_n \tag{30}$$

$$\bigvee_{l \in L_n} \left[\begin{array}{c} \hat{\mathrm{z}}_{n,l,m} = 1 \\ \hat{\mathrm{w}}_{\xi,t,n,l-1} = 0 \end{array} \right] \implies r_{\xi,t,i,m} = 0$$
$$\forall\, \xi \in \Xi,\ t \in T,\ n \in N, i \in I_m^r,\ m \in \tilde{M}_n \tag{31}$$

where Eqn. (30) guarantees that variables indicating the completion of replenishment ($\hat{\mathrm{w}}_{\xi,t,n,l}$) are activated following the order of priorities, and Eqn. (31) constraints replenishments according to the completion of levels that are hierarchically higher.

9.2. An MILP reformulation

We reformulate the logic for inventory management in parallel arrangements using mixed-integer constraints. The reformulation of constraints (25)-(26) is given by Eqns. (32)-(33).

$$\sum_{m \in \tilde{M}_n} \hat{\mathrm{z}}_{n,l,m} = 1 \qquad \forall\, n \in N,\ l \in L_n \tag{32}$$

$$\sum_{l \in L_n} \hat{\mathrm{z}}_{n,l,m} = 1 \qquad \forall\, n \in N,\ m \in \tilde{M}_n \tag{33}$$

The implication on replenishment completion (Eqn. (28)) can be reformulated according to Eqn. (34).

$$(1 - \hat{\mathrm{z}}_{n,l,m}) + \hat{\mathrm{x}}^{y}_{t,m} + \hat{\mathrm{u}}^{0}_{\xi,t,i} + \sum_{m' \in M^{in}_{i} \setminus R_n} \hat{\mathrm{g}}_{\xi,t,i,m'} + (1 - \hat{\mathrm{w}}_{\xi,t,l}) \geq 1$$
$$\forall\, \xi \in \Xi,\ t \in T,\ n \in N,\ l \in L_n,\ i \in I^{r}_{m},\ m \in \tilde{M}_n \tag{34}$$

where binary variable $\hat{\mathrm{u}}^{0}_{\xi,t,i}$ indicates if there is underutilization of unit i in scenario ξ at time period t. We enforce the definition of $\hat{\mathrm{u}}^{0}_{\xi,t,i}$ with the big-M constraint presented in Eqn. (35).

$$u_{\xi,t,i} \leq M\left(1 - \hat{\mathrm{u}}^{0}_{\xi,t,i}\right) \qquad \forall\, \xi \in \Xi,\ t \in T,\ i \in I \tag{35}$$

The condition (29) that indicates shortage of upstream material m in unit i can be reformulated with Eqns. (36)-(39),

$$\hat{\mathrm{x}}^{0}_{\xi,t,m} + (1 - \hat{\mathrm{g}}_{\xi,t,i,m}) \geq 1 \qquad \forall\, \xi \in \Xi,\ t \in T,\ i \in I,\ m \in M \tag{36}$$

$$\hat{\mathrm{v}}^{0}_{\xi,t,m} + (1 - \hat{\mathrm{g}}_{\xi,t,i,m}) \geq 1 \qquad \forall\, \xi \in \Xi,\ t \in T,\ i \in I,\ m \in M \tag{37}$$

$$\hat{\mathrm{u}}^{0}_{\xi,t,i'} + \sum_{m' \in M^{in}_{i'}} \hat{\mathrm{g}}_{\xi,t,i,m'} + (1 - \hat{\mathrm{g}}_{\xi,t,i,m}) \geq 1 \tag{38}$$

$$\forall\, \xi \in \Xi,\ t \in T,\ i \in I^{cons}_{m},\ i' \in I^{up}_{i,m},\ m \in M \tag{39}$$

where binary variable $\hat{\mathrm{v}}^{0}_{\xi,t,m}$ indicates if there is supply underutilization of material m in scenario ξ at time period t. We enforce the definition of $\hat{\mathrm{v}}^{0}_{\xi,t,m}$ with Eqn. (40),

$$v_{\xi,t,m} \leq M\left(1 - \hat{\mathrm{v}}^0_{\xi,t,m}\right) \qquad \forall\, \xi \in \Xi,\ t \in T,\ m \in M^S \tag{40}$$

Finally, the logic expressed in Eqns. (30)-(31) can be reformulated with Eqns. (41)-(42), respectively.

$$\hat{\mathrm{w}}_{\xi,t,n,l} \geq \hat{\mathrm{w}}_{\xi,t,n,l+1} \qquad \forall\, \xi \in \Xi,\ t \in T,\ n \in N,\ l \in L_n \tag{41}$$

$$\sum_{i \in I^r_m} r_{\xi,t,i,m} \leq M\left(1 - \hat{\mathrm{z}}_{n,m,l}\right) + M\hat{\mathrm{w}}_{\xi,t,n,l-1}$$
$$\forall\, \xi \in \Xi,\ t \in T,\ n \in N,\ l \in L_n, m \in \tilde{M}_n \tag{42}$$

where the parameter M is an upper bound for the total replenishment from all units $i \in I^r_m$.

It is important to remark that Eqns. (32)-(42) only represent one MILP reformulation of the logic developed for inventory management in parallel arrangements. Other reformulations with different number of variables and constraints are possible; they might lead to stronger or weaker formulations with respect to the LP relaxation.

10. Policy for inventories in series

The inventory planning for materials that undergo sequential transformation is based on multi-echelon inventory theory. We identify from the network structure processing paths (k) starting at raw material nodes and finishing at end product nodes; the purpose is to coordinate inventory management for the materials in these paths. A multi-echelon arrangement is a subset of storable materials ($\bar{M}_k \subseteq M^x$) associated with a particular processing path. We define an echelon as the subset ($\bar{M}_{k,e} \subseteq \bar{M}_k$) containing a number e of the most downstream materials in multi-echelon arrangement $\bar{M}_k$; echelons are numbered from the most downstream (echelon $\bar{M}_{k,1}$) to the most upstream (echelon $\bar{M}_{k,|E_K|}$), according to the conventions from multi-echelon literature. An illustration of the echelons comprising a multi-echelon arrangement is presented in Fig. 5.

10.1. Logic-based formulation

Formally, the subsets of materials in echelon $\bar{M}_{k,e}$ is given by Eqn. (43),

$$\bar{M}_{k,e} = \Big\{ m : m \in \left\{ \bar{M}_{k,e-1} \cup m_{k,e} \right\} \Big\} \tag{43}$$

where $m_{k,e}$ is the storable material preceding other $e-1$ materials in processing path k. Consequently, echelon $\bar{M}_{k,1}$ only contains one final product ($m_{k,1} \in M^D \ \ \forall\, k \in K$).

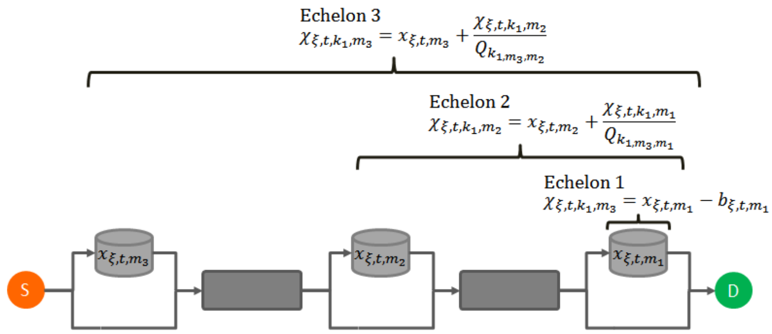


Figure 5: A multi-echelon arrangement with 3 echelons

The logic of basestock policies in multi-echelon systems is based on the concept of echelon inventory level. The echelon inventory level considers the available inventory for all the materials that belong to the echelon. The challenge to define the echelon inventory level in process networks is that materials change their identity through the production process; therefore, we have to consider the mass balance coefficients ($A_{i,m}$ and $B_{i,m}$) to calculate the equivalence between one material and its downstream successor. The inventory level ($\chi_{\xi,t,k,e}$) for echelons $\bar{M}_{k,1}$ and $\bar{M}_{k,e}$ can be calculated from Eqns. (44)-(45), respectively.

$$\chi_{\xi,t,k,1} = x_{\xi,t,m} - b_{\xi,t,m} \qquad \forall\ \xi \in \Xi,\ t \in T,\ k \in K,\ m = m_{k,1} \quad (44)$$

$$\chi_{\xi,t,k,e} = \frac{1}{Q_{k,e,e-1}} \chi_{\xi,t,k,e-1} + x_{\xi,t,m} \qquad \forall\ \xi \in \Xi,\ t \in T,\ k \in K,\ e \in E_k \backslash \{e = 1\},\ m = m_{k,e} \quad (45)$$

where $Q_{k,e,e-1}$ is the conversion ratio in the process that transforms material $m_{k,e}$ into material $m_{k,e-1}$ following processing path k. It is worth noticing that the inventory level of echelon $\bar{M}_{k,1}$ includes backorders, and that our process does not consider *in-transit* inventory since the transportation between units is assumed to be instantaneous.

Based on the echelon inventory level, we can extend the capacitated single-echelon basestock policy for inventory planning in sequential production processes. The idea is to define basestock levels ($y_{t,k,e}$) for each echelon, such that the available downstream inventory is considered in the replenishment decisions corresponding to material $m_{k,e}$. The logic for capacity utilization of the units ($i \in I_m^r$) that can replenish inventory $x_{\xi,t,m}$ remains the same as in the single-echelon system, except that we now have to consider the case in which underutilization is forced because upstream material shortage. In a multi-echelon arrangement, the conditions allowing backorders and underutilization are given by expressions (46)-(47), respectively.

$$\chi_{\xi,t,k,1} > 0 \implies b_{\xi,t,m} = 0 \qquad \forall\ \xi \in \Xi,\ t \in T,\ k \in K,\ m = m_{k,1} \tag{46}$$

$$[\chi_{\xi,t,k,e} < y_{t,k,e}] \ \wedge\ [g_{\xi,t,m} = 0\ \ \forall\ m \in M_i^{in}] \implies u_{\xi,t,i} = 0$$
$$\forall\ \xi \in \Xi,\ t \in T,\ k \in K,\ e \in E_k,\ i \in I^r_{m_{k,e}} \tag{47}$$

The equations defining echelon inventory levels (Eqns. (44)-(45)) and the logic controlling production decisions (Eqns. (46)-(47)) can be used in a logic-based formulation to find the optimal parameters of the basestock policy. For processing networks with multi-echelon arrangements, the parameters to optimize are the basestock levels for each echelon ($y_{t,k,e}$).

10.2. A MILP reformulation

We reformulate the logic for inventory management in multi-echelon arrangements using mixed-integer constraints. Similarly to the capacitated single-echelon basestock policy, this reformulation requires variables that indicate the state of the inventory level. We introduce binary variable $\hat{\chi}^0_{\xi,t,k,1}$ indicating if inventory in echelon 1 is empty, and variable and $\hat{\chi}^y_{\xi,t,k,e}$ indicating if inventory of echelon e is at basestock level. The definition for these variables is enforced with Eqns. (48)-(50).

$$\chi_{\xi,t,k,1} \leq M\left(1 - \hat{\chi}^0_{\xi,t,k,1}\right) \qquad \forall\ \xi \in \Xi,\ t \in T,\ k \in K \tag{48}$$
$$\chi_{\xi,t,k,e} \leq y_{t,k,e} \qquad \forall\ \xi \in \Xi,\ t \in T,\ k \in K, e \in E_k \backslash \{e = 1\} \tag{49}$$
$$\chi_{\xi,t,k,e} \geq y_{t,k,e} - M\left(1 - \hat{\chi}^y_{\xi,t,k,e}\right) \quad \forall\ \xi \in \Xi,\ t \in T,\ k \in K, e \in E_k \backslash \{e = 1\} \tag{50}$$

The implication presented in Eqn. (46), preventing stockouts if inventory is available, can be reformulated with big-M constraint (51).

$$b_{\xi,t,m} \leq M\hat{\chi}^0_{\xi,t,k,1} \qquad \forall\ \xi \in \Xi,\ t \in T,\ k \in K, m = m_{k,1} \tag{51}$$
$$\tag{52}$$

Finally, condition (47) can be reformulated with (47).

$$\hat{\chi}^y_{\xi,t,k,e} + \sum_{m \in M_i^{in}} \hat{g}_{\xi,t,i,m} + \hat{u}^0_{\xi,t,i} \geq 1 \quad \forall\ \xi \in \Xi,\ t \in T,\ k \in K, e \in E_k, i \in I^r_{m_{k,e}} \tag{53}$$

The MILP reformulation proposed for multi-echelon arrangements is obtained by enforcing Eqns. (44)-(45) together with Eqns. (48)-(53). The resulting reformulation is only one reformulation of the logic proposed for multi-echelon arrangements. Other reformulations are also possible.

11. Evaluating inventory planning strategies with closed-loop Monte Carlo simulations

In order to asses the potential benefits of implementing a policy-based production planning, we compare the planning decisions obtained by solving the logic-based SP formulation with the decision obtained from the equivalent two-stage SP formulation. The challenge for large-scale problem is that the number of scenarios in multiperiod problems grows exponentially; therefore, we cannot calculate the REV exactly as we have done with the motivating example in Section 7. The alternative is to use the planning strategies in a receding horizon with the purpose of simulating the sequential implementation of the decision-making process. The proposed closed-loop Monte Carlo simulations resemble Economic MPC [34], but our focus is on finite planning horizons and we solve a stochastic programming problem at each time period.

The scenarios for the SP formulations represent possible values of the exogenous uncertain parameters, from the current period until the end of the planning horizon. We assume to have a probabilistic description of these parameters, which allow us generating possible trajectories using sampling techniques. The multiperiod SP formulations with sampled scenarios can be considered sample-path optimization problems [35]; the purpose of solving these sample-path problems is to estimate the optimal planning strategy based on a reduced set of scenarios.

Four different parameters must be specified for the implementation of the closed-loop simulations: number of replications, length of the simulation horizon, length of the planning horizon, and sample size for the planning problem. The number of replications specifies how many closed-loop simulations we run; a large number of replications is desirable because it allows better estimation of the simulation expected cost and its variance. The simulation horizon is the length of the simulation and specifies how many optimization problems we solve in each replication. The planning horizon is the length of the sample-paths used as scenarios in the multiperiod formulations; it defines how far into the future we look when solving the planning problem. Finally, the sample size specifies how many scenarios we include in the optimization problems; a larger the number of scenarios tends to produce better approximations of the full problem, but the sample size is constrained by the computational complexity of the instances.

The closed-loop simulations are used to evaluate the performance of the proposed formulations for inventory and production planning. The procedure to estimate the expected performance of these planning strategies has the following steps:

1. Establish the parameters for the closed-loop simulations.
2. Start a closed-loop simulation ($t^* = 1$).
   - 2.1. Observe the state at simulation time (t^*).
   - 2.2. Generate scenarios by randomly sampling paths of the exogenous uncertain parameters.

2.3. Formulate and solve the stochastic optimization problem.

2.4. Implement the decisions corresponding to the current simulation time (t^*).

2.5. Randomly generate the realizations of the exogenous uncertain parameters for the next time period in the simulation ($t^* + 1$).

2.6. Roll the simulation time forward ($t^* = t^* + 1$).

2.7. If simulation time is less than the simulation horizon, go back to Step 2.1. Otherwise, continue to Step 3.

3. If the number of closed simulations is less than the number of replications, go back to Step 2. Otherwise, continue to Step 4.

4. Calculate the statistics over all replications and terminate.

Fig. 6 shows the trajectory of uncertain parameters in a closed-loop Monte Carlo simulation, where the past is represented by a unique path and the future is represented by alternative paths indicating possible scenarios. The simulation presented in Fig. 6 is performed over five periods ($t^* = 0$ to $t^* = 4$). At each period, a stochastic SP problem with a 4-period planning horizon is solved. Then, time moves forward and uncertainty is revealed.

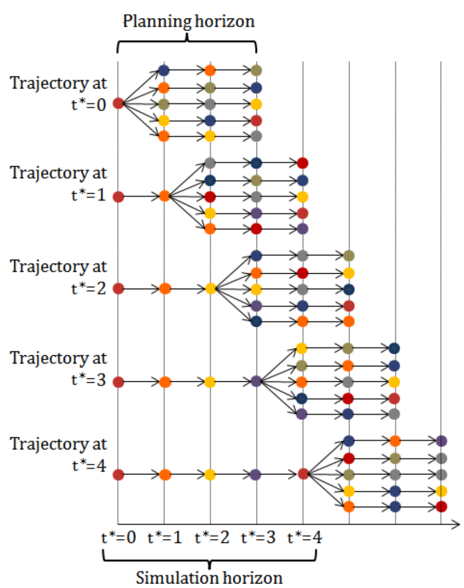


Figure 6: Trajectory of uncertain parameters in a closed-loop Monte Carlo simulation.

In the following examples, we compare the performance of the planning decisions obtained from the logic-based SP formulations and the equivalent two-stage SP formulations. For both of them, we use exactly the same sampled scenarios in every instance. In addition, we use the same realizations of the uncertain parameters in the implementation of the closed-loop simulations. The mathematical models and the sampling procedure are implemented in AIMMS 4.8.3; all optimization problems are solved using GUROBI 6.0.0 on an Intel Core i7 CPU 2.93 Ghz processor with 4 GB of RAM.

12. Example with inventories in parallel

This example illustrates the implementation of the two-stage and the priority policy approaches for inventory planning in production processes with parallel arrangements. The structure of the network is shown in Fig. 7. The purpose of the network is to transform raw material (m_0) into four products ($m_1, m_2, m_3,$ and m_4) with final demands. The units producing final products share the same raw material, which creates competition for the replenishment of inventories. The process network only allows storage of final products.

Raw material supply and final product demands are deterministic. Supply (S_{t,m_0}) is constant at 90 *ton/period* throughout the time horizon. Demand ($D_{t,m}$) for final products is deterministic but time-varying. The demand profiles are presented in Fig. 8.

Mass balance coefficients ($A_{i,m}$) indicating the amount of materials consumed per unit production rate are presented in Table 2; all mass balance coefficients for the amount of material produced per unit production rate are equal to one ($B_{i,m} = 1 \;\; \forall\, i \in I_m^{prod},\; m \in M_i^{out}$). Unit holding costs ($H_{t,m}$) and unit backorder costs ($P_{t,m}$) are constant in time; their values given in Table 3.

Unit	A_{i,m_0}
i_1	1.180
i_2	1.355
i_3	0.724
i_4	0.570

Table 2: Consumption coefficients (A_{i,m_0}) for the example with inventories in parallel.

Material	$H_{t,m}$ [\$ ton/period]	$P_{t,m}$ [\$ ton/period]
m_1	0.55	4.40
m_2	0.45	3.60
m_3	0.65	5.20
m_4	0.85	6.80

Table 3: Cost parameters for the example with inventories in parallel.

Available production capacities ($C_{\xi,t,i}$) in the processing units are consid-

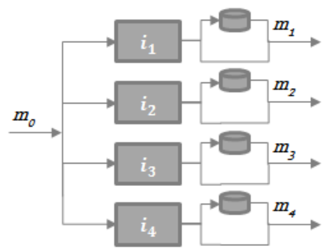


Figure 7: Structure of the example with an arrangement of parallel inventories.

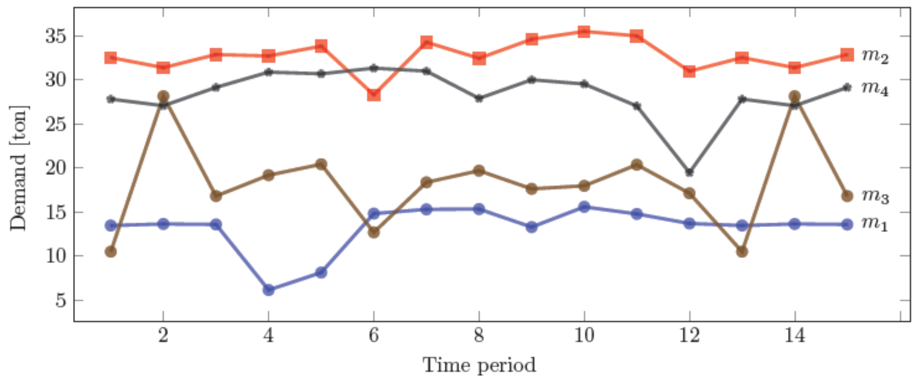


Figure 8: Deterministic demands for the example with inventories in parallel.

ered uncertain. Each uncertain parameter is modeled as an independent time-homogeneous Discrete Time Markov Chain (DTMC) with the purpose of describing the state-dependent evolution of uncertainty in industrial processes. The states of the DTMCs characterize the value of the uncertain parameters; each parameter has three states that imply different available production capacities. Table 4 shows the value of each uncertain parameter according to their state.

Parameter	State		
	Low	*Nominal*	*High*
C_{ξ,t,i_1}	13.23	14.70	16.17
C_{ξ,t,i_2}	32.13	35.70	39.27
C_{ξ,t,i_3}	22.68	25.20	27.72
C_{ξ,t,i_4}	28.35	31.50	34.65

Table 4: Production capacities according to their DTMC state for the example with inventories in parallel.

We assume that all uncertain parameters are initially at their nominal values. The evolution of each DTMCs is characterized with the one-step transition matrix (Π). The same transition matrix is used to model the evolution of all production capacities. The transition matrix is given by Eqn. (54).

$$\Pi = \begin{array}{c} \begin{array}{ccc} Low & Nominal & High \end{array} \\ \left[\begin{array}{ccc} 0.70 & 0.25 & 0.05 \\ 0.15 & 0.70 & 0.15 \\ 0.05 & 0.25 & 0.70 \end{array}\right] \end{array} \begin{array}{l} \\ Low \\ Nominal \\ High \end{array} \tag{54}$$

It is worth noticing that in a single time period, there are 4 uncertain parameters with 3 possible outcomes, giving rise to 81 possible combinations. In a multiperiod problem with 6 time periods there are millions (81^6) of possible scenarios, which would result in an intractable optimization problem for any practical purpose.

We compare performance of the two-stage SP and the logic-based SP inventory planning strategies based on **25 closed-loop simulations**. At each period, each strategy solves a stochastic optimization problem with **10 sampled scenarios** and a **planning horizon of 6 time periods**. In the logic-based SP formulation, we enforce the priority policy for the parallel arrangement made up by the four final products ($\tilde{M}_1 = \{m_1, m_2, m_3, m_4\}$) in planning periods 2, 3, 4, and 5. It is unnecessary to enforce the policy in the first planning period because the uncertainty has already been revealed; enforcing the policy in the last planning period does not bring any benefit because no future periods can be anticipated. The length of the **simulation horizon is set to 12 periods**.

Table 5 presents the computational statistics for the two-stage SP formulation and the MILP reformulation of the logic-based SP model. The number of variables and constraints remain the same throughout the simulations because we use a receding horizon approach. All MILPs are solved to an optimality gap of 0.25%.

Statistic	Formulation	
	Two-stage SP	Logic-based SP
Constraints:	1,812	4,204
Continuous variables:	2,460	2,484
Binary variables:	0	976
Instances solved to optimality:	300	300
Mean CPU time of instances [s]:	< 1	176 ($\pm$ 501)

Table 5: Computational statistics of the two-stage SP and the logic-based SP formulations for the example with inventories in parallel.

Table 5 shows a significant difference in computational complexity of both models. It is important to remark that the two-stage SP formulation is strictly a relaxation of the logic-based SP formulation, and it has only a subset of the variables and constraints. As a consequence, the mean CPU time required to solve the instances of the two-stage SP model is less than one second; the mean CPU time for the instances of the logic-based SP model is 176 seconds, with a standard deviation of 501 seconds.

The results of the closed-loop simulations can be observed in Figs. 9-10, where the shaded lines represent the cost trajectories for the individual replications and the solid lines are the averages over all replications. The figures show similar costs for both approaches, with a slightly higher stockout cost for the two-stage SP model that can be observed in periods 10 and 11. The trajectories presented in Figs. 9-10 evidence significant variability in the results obtained from the implementation of both planning strategies. This variability is inherent to the nature of the problem, because uncertainty in production capacities constitutes a high risk for stockouts. The main performance metric for the planning strategies is the expected cost of simulations. Table 6 presents the mean cost for each planning strategy over all simulations, together with its standard deviation and the corresponding service level (type β).

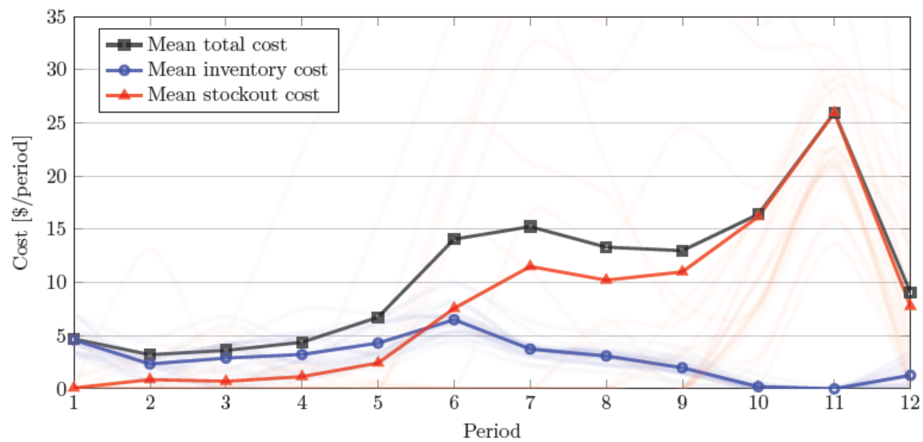


Figure 9: Trajectories of holding and stockout costs obtained from the two-stage SP formulation for the example with inventories in parallel.

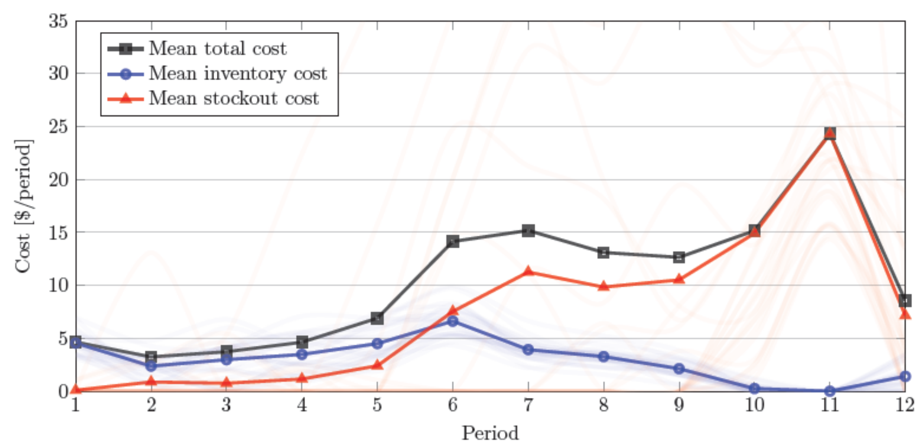


Figure 10: Trajectories of holding and stockout costs obtained from the logic-based SP formulation for the example with inventories in parallel.

The results from Table 6 show a 2.7% reduction in the total expected cost for the logic-based SP model in comparison to the two-stage SP model; the reduction is obtained from lower stockout costs without increasing the inventory cost significantly. Although the difference is rather small, the results suggest that the logic-based SP model is more effective at selecting the materials that are stored as inventories according to the representation of the future given by the scenarios. Despite the large variability in the simulation total costs, we can be confident in the advantages of the logic-based SP model because it consistently outperforms the two-stage SP model throughout the replications. A comparison

Metric	Model	
	Two-stage SP	Logic-based SP
Mean inventory cost [$]:	34.26	34.50
Mean stockout cost [$]:	95.54	90.80
Mean total cost [$]:	129.80	126.30
Standard deviation [$]:	110.34	110.69
Service level (β):	0.985	0.986

Table 6: Performance metrics for the two-stage SP and the logic-based SP formulations.

of the total cost for each replication is presented in Fig 11; we observe that the two-stage SP model only outperforms the logic-based SP model by a negligible amount in 8 out 25 replications.

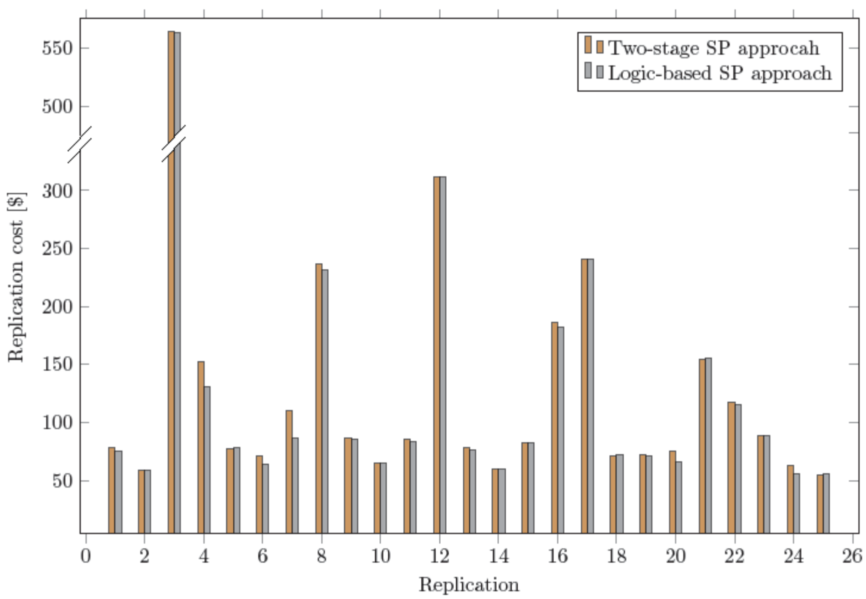


Figure 11: Comparison of replication costs for the example with inventories in parallel.

13. Example with inventories in series

We use this example to compare the inventory plan obtained from the two-stage SP formulation with the plan dictated by the logic-based SP formulation modeling the multi-echelon inventory policy. The example has been adapted from the Example 1 presented by Terrazas-Moreno et al. [48], and originally proposed by Straub and Grossmann [43]. The purpose of the process network is to transform a single raw material (m_0) into one final product (m_3). The network offers three alternative processing paths, from which we identify one multi-echelon arrangement including both storable materials: $\bar{M}_1 = \{m_1, m_3\}$. The structure of the network and the echelons of $\bar{M}_1$ are shown in Fig. 12.

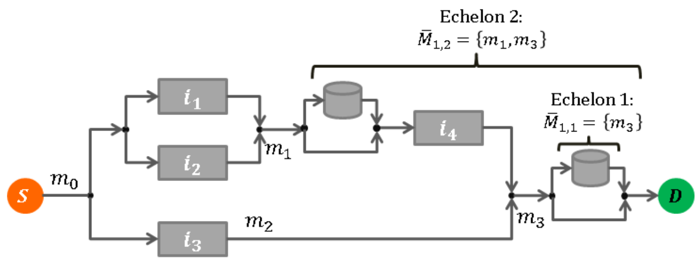


Figure 12: Structure of the example with a multi-echelon arrangement of inventories.

Supply availability (S_{ξ,t,m_0}), available production capacities ($C_{\xi,t,i}$), and demand are considered uncertain. Supply and demand are modeled as normally distributed random variables; their mean values ($\bar{S}_{\xi,t,m_0}$, $\bar{D}_{\xi,t,m_3}$) are periodic functions presented in Fig. 13; their coefficients of variation are set to 15%.

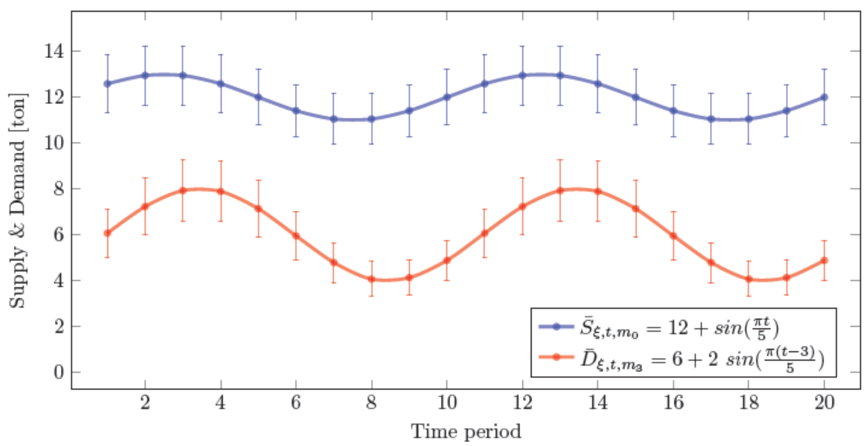


Figure 13: Time-varying normally distributed supply and demand for the example with inventories in series.

The capacity ($C_{\xi,t,i}$) of each processing unit is modeled as an independent time-homogeneous Discrete Time Markov Chain (DTMC) with the purpose of describing probabilistic failures. The DTMCs characterize the states of the units that can be either working normally (*up*) or failed (*down*). Table 7 shows the value of production capacities according to their state.

Parameter	State	
	Up	*Down*
C_{ξ,t,i_1}	5	0
C_{ξ,t,i_2}	5	0
C_{ξ,t,i_3}	7	0
C_{ξ,t,i_4}	9	0

Table 7: Production capacities according to the state of units for the example with inventories in series.

We assume that all units are initially at their *up* state. The evolution of each DTMCs is characterized with the one-step transition matrices (Π_i) given by Eqn. (55).

$$\Pi_{i_1} = \begin{array}{cc} \begin{array}{cc} Up & Down \end{array} & \\ \left[\begin{array}{cc} 0.97 & 0.03 \\ 0.50 & 0.50 \end{array}\right] & \begin{array}{c} Up \\ Down \end{array} \end{array} \qquad \Pi_{i_2} = \begin{array}{cc} \begin{array}{cc} Up & Down \end{array} & \\ \left[\begin{array}{cc} 0.95 & 0.05 \\ 0.50 & 0.50 \end{array}\right] & \begin{array}{c} Up \\ Down \end{array} \end{array}$$

$$\Pi_{i_3} = \begin{array}{cc} \begin{array}{cc} Up & Down \end{array} & \\ \left[\begin{array}{cc} 0.96 & 0.04 \\ 0.50 & 0.50 \end{array}\right] & \begin{array}{c} Up \\ Down \end{array} \end{array} \qquad \Pi_{i_2} = \begin{array}{cc} \begin{array}{cc} Up & Down \end{array} & \\ \left[\begin{array}{cc} 0.93 & 0.07 \\ 0.50 & 0.50 \end{array}\right] & \begin{array}{c} Up \\ Down \end{array} \end{array} \tag{55}$$

From the individual states of the processing units, we know that there are 16 different discrete states for the entire system. The entire system could be characterize as a single DTMC, but it is unnecessary because we assume that the state transitions for each unit only depend on its own state. In addition to the discrete states characterizing production capacities, supply and demand are modeled with continuous distributions; therefore, the total number of scenarios is uncountable.

The remaining parameters of the example are deterministic; they are given in Tables 8 and 9. Mass balance coefficients indicating amount of material produced per unit production rate are all equal to one ($B_{i,m} = 1 \;\; \forall \, i \in I_m^{prod}, \; m \in M_i^{out}$). Unit holding costs ($H_{t,m}$) and unit backorder costs ($P_{t,m}$) are constant in time.

Unit	Material	
	A_{i,m_0}	A_{i,m_1}
i_1	1.087	-
i_2	1.111	-
i_3	1.176	-
i_4	-	1.333

Table 8: Consumption coefficients ($A_{i,m}$) for the example with inventories in series.

Material	$H_{t,m}$ [$ ton/period]	$P_{t,m}$ [$ ton/period]
m_1	1	-
m_3	3	10

Table 9: Cost parameters for the example with inventories in series.

We compare performance of the two-stage SP and the logic-based SP inventory planning strategies based on **25 closed-loop simulations**. At each period, both strategies solve a stochastic optimization problem with **10 sampled scenarios** and a **planning horizon of 5 time periods**. In the logic-based SP formulation, we enforce the multi-echelon basestock policy for arrangement $\bar{M}_1$. The length of the **simulation horizon is set to 15 periods**.

The computational statistics for the instances of each formulation are presented in Table 10. All MILPs are solved to an optimality gap of 0.25%.

Statistic	Formulation	
	Two-stage SP	Logic-based SP
Constraints:	1,304	2,204
Continuous variables:	1,750	1,760
Binary variables:	0	450
Instances solved to optimality:	300	300
Mean CPU time of instances [s]:	< 1	189 ($\pm$ 487)

Table 10: Computational statistics of the two-stage SP and the logic-based SP formulations for the example with inventories in series.

The cost trajectories for the two-stage SP and the logic-based SP models are presented in Figs. 14-15, respectively. Shaded lines represent the trajectories for individual replications and solid lines are the averages. The figures show a trend for the two-stage SP model to produce higher inventory costs; this can be observed at periods 3, 4, and 5.

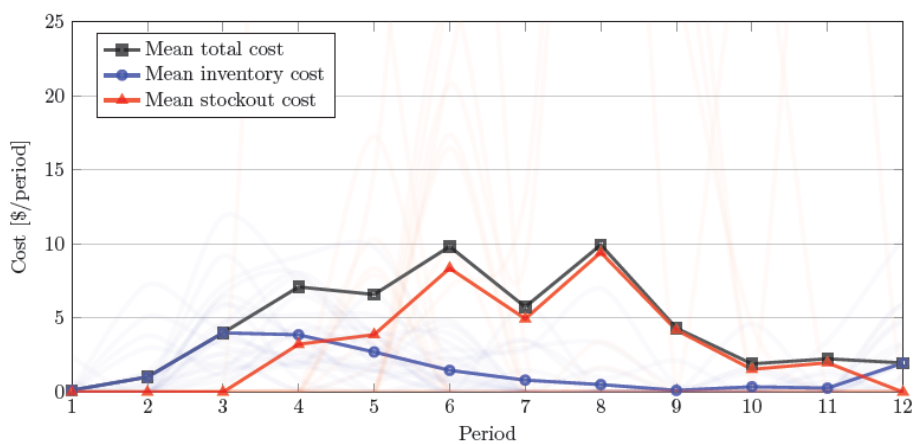


Figure 14: Trajectories of holding and stockout costs obtained from the two-stage SP formulation for the example with inventories in series.

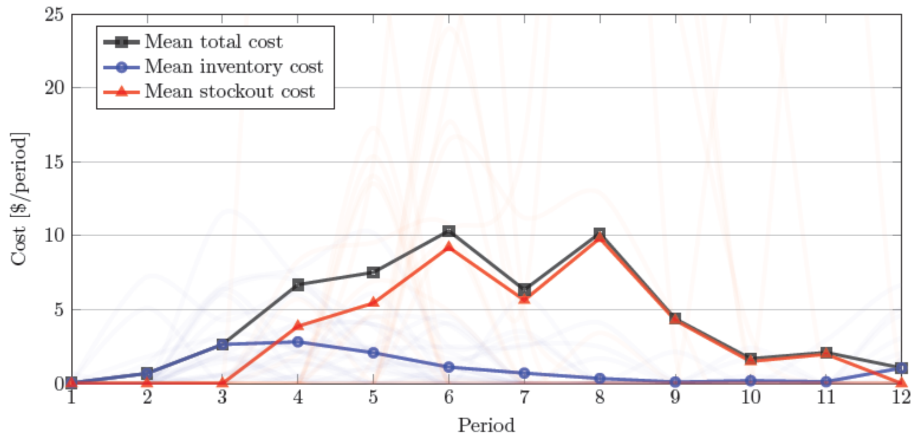


Figure 15: Trajectories of holding and stockout costs obtained from the logic-based SP formulation for the example with inventories in series.

The trajectories in Figs. 14-15 show significant variability for holding and stockout cost across replications. Variability in this process network is the result of random failures that produce high stockout risk. The performance metrics for the planning models are presented in Table 11.

Metric	Model	
	Two-stage SP	Logic-based SP
Mean inventory cost [$]:	32.30	21.07
Mean stockout cost [$]:	42.33	48.31
Mean total cost [$]:	74.62	69.38
Standard deviation [$]:	59.10	56.34
Service level (β):	0.958	0.952

Table 11: Performance metrics for the two-stage SP and the logic-based SP formulations.

We observe from Table 11 a reduction in the mean total cost obtained from the logic-based SP model that correspond to 7.20% of the cost obtained from the two-stage SP model. The reduction is the result of an inventory planning strategy that is more effective at balancing holding and backorders cost; the logic-based SP model benefits from an increased coordination between intermediate and final product inventory levels.

Finally, in Fig 16 we present the cost obtained for each replication using the two-stage SP and the logic-based SP planning strategies. The figure shows that the logic-based SP approach yields a lower cost than the two-stage SP approach in 19 out of 25 replications. These results clearly illustrate the advantages of the multi-echelon basestock policy for inventory planning in process networks.

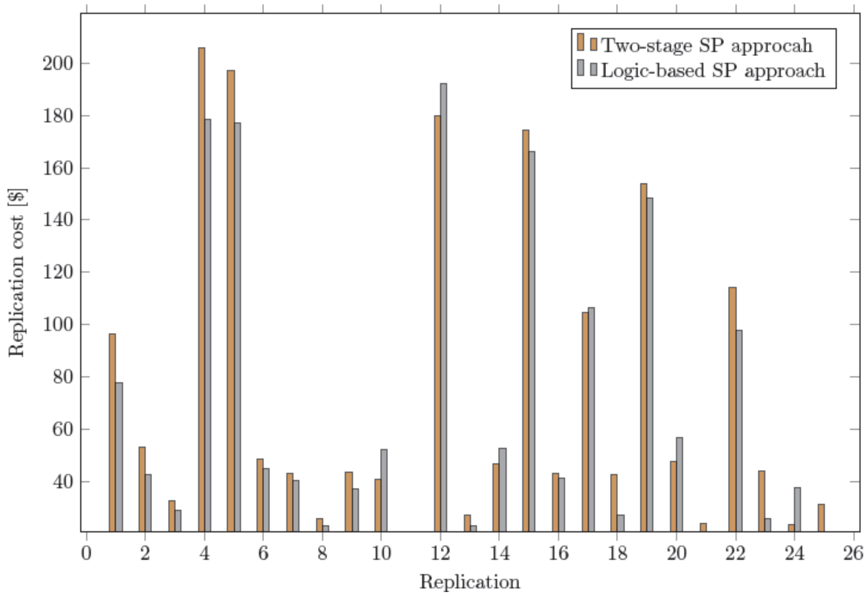


Figure 16: Comparison of replication costs for the example with inventories in series.

14. Conclusions

In this article, we have proposed a policy-based approach for the stochastic inventory planning problem in process networks. Our motivation originates from the effectiveness of policies for inventory management and their appeal for industrial implementation. Given the difficulty to obtain optimal policies analytically in process networks, we have developed the logic describing these policies with the purpose of including them into the production and inventory planning problem.

We have proposed two sets of logic rules for inventory planning in networks with parallel and sequential structures. The logic is formulated as a GDP model that avoids anticipativity in stochastic programing problems and yields the optimal parameters of inventory policies. We implemented the MILP reformulations of our logic-based SP models in two examples, and compared the results with the corresponding two-stage SP models. The comparisons were based on closed-loop simulations that resemble the actual implementation of these planning strategies in an industrial environment. Despite the increase in the computational complexity of the instances, the examples show a significant improvement in the inventory plans obtained from the logic-based SP model.

The proposed logic-based SP formulation has the advantage of being completely flexible with respect to the probabilistic description of the uncertain parameters. The only requirement for the model is to be able to generate scenarios describing the evolution of uncertain parameters by any forecasting method. This feature is specially important for industrial applications in which correlation and autocorrelation of the uncertain parameters is very common, and allows using historical data in the inventory planning model.

The logic developed for inventory planning in process networks with parallel and sequential structures can be extended to address networks of arbitrary topology with complex uncertainty models. There is an extraordinary potential for inventory optimization in these networks because their complexity conceals the most effective planning strategies. This contribution offers a novel approach for a very challenging problem particular to the process industry.

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