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Global Optimization Algorithm for Capacitated Multi-facility Continuous Location-Allocation Problems

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Abstract In this paper we propose a nonlinear Generalized Disjunctive Programming (GDP) model to optimize the 2-dimensional continuous location and allocation of the potential facilities based on their maximum capacity and the given coordinates of the suppliers and customer points. The model belongs to the class of Capacitated Multi-facility Weber Problem. We propose a bilevel decomposition algorithm that iteratively solves a discretized MILP version of the model, and its nonconvex NLP for a fixed selection of discrete variables. Based on the bounding properties of their subproblems, ϵ-convergence is proved for this algorithm. We apply the proposed method to random instances varying from 2 suppliers and 2 customers to 40 suppliers and 40 customers, from one type of facility to 3 different types, and from 2 to 32 potential facilities. The results show that the algorithm is more effective at finding global optimal solutions than general purpose global optimization solvers tested.

Keywords location-allocation problem · Weber problem · nonconvex optimization · generalized disjunctive programming · mixed-integer nonlinear programming

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This paper is dedicated to the memory of Professor Chris Floudas.

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1 Introduction

In this paper, we address the design of networks that involves the selection and location of facilities in the continuous 2-dimensional space. The problem is formulated as a continuous facility location-allocation problem with limited capacity, also known as the Capacitated Multi-facility Weber Problem (CMWP) with fixed costs. The objective of this type of problem is to determine locations in continuous 2-dimensional space for opening new facilities that are connected to supply and customer nodes, taking into account limited capacities and transportation costs [4].

The Weber problem was named after Alfred Weber [23], whose work is considered to have established the foundations of modern location theories. In his first problem, he considered one facility to be located based on two supplier and one customer, when these three points are not collinear [23,12]. The basic model assumes Euclidean distances, but other distance functions such as Manhattan (or ℓ_1) norm and ℓ_p norm have also been used depending on the application and region where the transportation is considered.

The capacitated version of the Weber problem, known as Capacitated Multi-facility Weber problem (CMWP), considers that the facilities to be installed have a maximum capacity. As shown by Sherali and Nordai [19], this class of problems is NP-hard even if all the fixed points are located on a straight line. The general formulation for a CMWP is as follows.

$$\min \Phi = \sum_{i,j} c_{ij} \cdot f_{ij} \cdot \left\{(x_i - x_j)^2 + (y_i - y_j)^2\right\}^{1/2} \tag{1a}$$

$$s.t. \quad \sum_{j} f_{ij} = a_i \qquad \forall\, i \in \mathcal{I} \tag{1b}$$

$$\sum_{i} f_{ij} = d_j \qquad \forall\, j \in \mathcal{J} \tag{1c}$$

$$f_{ij} \geq 0 \qquad \forall\, i \in \mathcal{I},\ j \in \mathcal{J} \tag{1d}$$

$$x_i, y_i \geq 0 \qquad \forall\, i \in \mathcal{I} \tag{1e}$$

where i is the index of the facilities to be located, j is the index of customers, (x_j, y_j) are the fixed coordinates of the customer j, a_i is the capacity of facility i, d_j is the demand of customer j, and c_{ij} is the cost of unit flow per unit distance from facility i to customer j. The decision variables are: (x_i, y_i), which represent the coordinates of a new facility i; and f_{ij}, which is the material flow between facility i to customer j.

Cooper [6] was the first to attempt solving this type of location-allocation problem in 1972. He proposed exact and approximate solution methods based on the property that an optimal allocation occurs at the extreme point of the transportation polytope, while the optimal set of locations lies on the convex hull of the locations of the existing facilities. His exact formulation requires the explicit enumeration of the extreme points of the transportation polytope, limiting its application to small problems. Coopers heuristic approach, known as the Alternating Transportation-Location (ATL) method, exploits the structure of the problem by alternating the solution of the transportation and allocation problems until convergence is achieved, but there is no guarantee of global optimality. His ATL heuristic is further developed in [7,8], extending it to CMWP with fixed charges.

Sherali and Shetty [17] develop a cutting plane algorithm for the rectilinear distance location-allocation problem. Sherali and Tuncbilek [20] propose a branch-and-bound algorithm for the squared-Euclidean distance location-allocation problem. Sherali, Al-Loughani, Subramanian [18] developed a branch-and-bound algorithm based on the partitioning of the allocation space that finitely converges to a global optimum within a tolerance. Chen, Pan, Ko [5] reformulated the problem as a sequence of nonlinear second order conic problems, and applied the semi-smooth Newton method to solve it.

Apart from the exact methods, there are several heuristic methods that have been applied to this problem [4,3]. Akyuz, Oncan and Altinel [1] use Lagrangean relaxation and the subgradient method to develop a heuristic for the multi-commodity CMWP. Luis, Salhi and Nagy [14] report a new variant of the CMWP with fixed costs for opening facilities and propose a constructive heuristic based on the concept of restricted regions and greedy randomized adaptive search procedure (GRASP).

In this paper, we propose an extension of the CMWP that considers fixed cost for opening new facilities, and two sets of fixed-location points: suppliers i and customers j, as represented in Figure 1. The latter goes back to the original Weber problem, in which the location of the facility had to be determined in relation to 2 suppliers and 1 customer points. The model is a nonconvex Mixed-Integer Nonlinear Programming (MINLP), in which the nonconvexity comes from the variable multiplication in the transportation cost. This is, to the best of our knowledge, an original problem not reported before that high practical applicability [13].

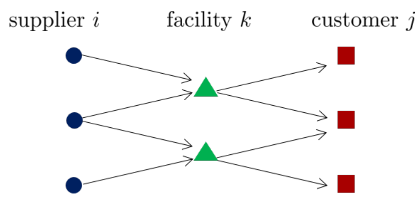

Fig. 1 Representation of the nodes in the network

The remainder of the paper is organized as follows. We begin by presenting in Section 2 the problem statement, its General Disjunctive Programming (GDP) formulation, and its reformulation as a nonconvex MINLP. In Section 3 we propose a global optimization algorithm based on the partitioning of the space, which is guaranteed to have ϵ-convergence. In Section 4 we introduce a small test problem. The performance of the algorithm is assessed in Section 5 by solving randomly generated instances with the proposed algorithm and comparing the solution, optimality gap and computational time with general purpose global optimization solvers.

2 Problem statement

Given is a set of suppliers $i \in \mathcal{I}$, with their respective fixed locations (x_i, y_i), availability a_i, and cost of material supply cs_i. Given is also a set of customers $j \in \mathcal{J}$, with their respective fixed locations (x_j, y_j), and demands d_j. Given are the fixed and variable costs (ff_k and vf_k, respectively) of potential facilities $k \in \mathcal{K}$ with N different types, which are divided in subsets $\mathcal{K}_n \ \forall\ n = \{1, ..., N\}$ such that $\bigcup_n \mathcal{K}_n = \mathcal{K}$. The corresponding maximum capacity, mc_k, and conversion to product flows, cv_k, of these potential facilities are also known. Given are also the transportation costs between suppliers and facilities, and facilities and customers ($ft_{i,k}, ft_{k,j}$: fixed costs; $vt_{i,k}, vt_{k,j}$: variable costs). The problem is to find the optimal network of facilities (number, types, location, and corresponding flows) that minimizes the total cost.

The variables in the problem are the coordinates of potential facilities, (x_k, y_k), the distances between supplier and facility, $D_{i,k}$, and between facility and customer, $D_{k,j}$, the flows between supplier and facility, $f_{i,k}$, and between facility and customer, $f_{k,j}$, and the amount produced by each facility, f_k. There are also Boolean variables W_k (true if facility is built; false otherwise); $Z_{i,k}$ (true if material supply is transported between supplier and facility; false otherwise); and $Z_{k,j}$ (true if product is transported between facility and customer; false otherwise). The GDP [22] formulation is given by Equations (2a)-(2m).

$$\min\ \Phi = \sum_k Cost_k + \sum_i \sum_k Cost_{i,k} + \sum_k \sum_j Cost_{k,j} \tag{2a}$$

$$\begin{bmatrix} W_k \\ Cost_k = ff_k + vf_k \cdot f_k \\ 0 \le f_k \le mc_k \\ 0 \le x_k \le x_k^{\mathrm{U}} \\ 0 \le y_k \le y_k^{\mathrm{U}} \end{bmatrix} \vee \begin{bmatrix} \neg W_k \\ Cost_k = 0 \\ f_k = 0 \\ x_k = 0 \\ y_k = 0 \end{bmatrix} \quad \forall\ k \in \mathcal{K} \tag{2b}$$

$$\begin{bmatrix} Z_{i,k} \\ Cost_{i,k} = cs_i \cdot f_{i,k} + ft_{i,k} + vt_{i,k} \cdot f_{i,k} \cdot D_{i,k} \\ 0 \le f_{i,k} \le f_{i,k}^{\mathrm{U}} \\ D_{i,k}^{\mathrm{L}} \le D_{i,k} \le D_{i,k}^{\mathrm{U}} \end{bmatrix} \vee \begin{bmatrix} \neg Z_{i,k} \\ Cost_{i,k} = 0 \\ f_{i,k} = 0 \end{bmatrix} \quad \forall\ i \in \mathcal{I}, k \in \mathcal{K} \tag{2c}$$

$$\begin{bmatrix} Z_{k,j} \\ Cost_{k,j} = ft_{k,j} + vt_{k,j} \cdot f_{k,j} \cdot D_{k,j} \\ 0 \le f_{k,j} \le f_{k,j}^{\mathrm{U}} \\ D_{k,j}^{\mathrm{L}} \le D_{k,j} \le D_{k,j}^{\mathrm{U}} \end{bmatrix} \vee \begin{bmatrix} \neg Z_{k,j} \\ Cost_{k,j} = 0 \\ f_{k,j} = 0 \end{bmatrix} \quad \forall\ k \in \mathcal{K}, j \in \mathcal{J} \tag{2d}$$

$$D_{i,k} \geq \sqrt{(x_i - x_k)^2 + (y_i - y_k)^2} \qquad \forall\ i \in \mathcal{I}, k \in \mathcal{K} \tag{2e}$$

$$D_{k,j} \geq \sqrt{(x_j - x_k)^2 + (y_j - y_k)^2} \qquad \forall\ k \in \mathcal{K}, j \in \mathcal{J} \tag{2f}$$

$$W_k \iff \bigvee_i Z_{i,k} \qquad \forall\ k \in \mathcal{K} \tag{2g}$$

$$W_k \iff \bigvee_j Z_{k,j} \qquad \forall\ k \in \mathcal{K} \tag{2h}$$

$$\sum_k f_{i,k} \leq a_i \qquad \forall\ i \in \mathcal{I} \tag{2i}$$

$$\sum_i f_{i,k} \cdot cv_k = f_k \qquad \forall\ k \in \mathcal{K} \tag{2j}$$

$$f_k = \sum_j f_{k,j} \qquad \forall\ k \in \mathcal{K} \tag{2k}$$

$$\sum_k f_{k,j} = d_j \qquad \forall\ j \in \mathcal{J} \tag{2l}$$

$$W_k, Z_{i,k}, Z_{k,j} \in \{True, False\} \qquad \forall\ i \in \mathcal{I}, k \in \mathcal{K}, j \in \mathcal{J} \tag{2m}$$

The objective function (2a) includes costs for the facilities and for the transportation from the suppliers and to the customers. Disjunction (2b) determines the selection of facilities (W_k), while the disjunctions (2c) and (2d) determine the transportation links $\{i,\ k\}$ and $\{k,\ j\}$ with the corresponding Boolean variables ($Z_{i,k},\ Z_{k,j}$). Constraints (2e) and (2f) represent the Euclidean distances between suppliers and facilities, and facilities and customers, while the logic relations in (2g) and (2h) establish the existence of links depending on the choice of the facilities and vice-versa. Finally, constraints (2i)-(2l) define the mass balances as well as the availabilities and demands.

The model (2) is a nonconvex GDP due to the bilinear terms ($f \cdot D$) in the transportation cost, as can be seen in disjunctions (2c)-(2d). Equations (2e)-(2f) are nonlinear convex constraints since they correspond to Euclidean norms [2]. The presence of nonconvexities was the main motivation for representing the problem as a GDP. By having the bilinear terms as part of the disjunctions, the transportation costs are calculated only for the selected connections within an iterative procedure.

The GDP can be transformed into an MINLP using the hull reformulation, which yields the tightest relaxation for each disjunction [22]. Since the disaggregated variables can be reformulated back to the original variables, the resulting MINLP is given by Equations (3a)-(3v).

$$\min\ \Phi = \sum_k Cost_k + \sum_i \sum_k Cost_{i,k} + \sum_k \sum_j Cost_{k,j} \tag{3a}$$

$$Cost_k = ff_k \cdot w_k + vf_k \cdot f_k \qquad \forall\ k \in \mathcal{K} \tag{3b}$$

$$Cost_{i,k} = cs_i \cdot f_{i,k} + ft_{i,k} \cdot z_{i,k} + vt_{i,k} \cdot f_{i,k} \cdot D_{i,k} \quad \forall\ i \in \mathcal{I}, k \in \mathcal{K} \tag{3c}$$

$$Cost_{k,j} = ft_{k,j} \cdot z_{k,j} + vt_{k,j} \cdot f_{k,j} \cdot D_{k,j} \qquad \forall\ k \in \mathcal{K}, j \in \mathcal{J} \tag{3d}$$
$$D_{i,k} \geq \sqrt{(x_i - x_k)^2 + (y_i - y_k)^2} \qquad \forall\ i \in \mathcal{I}, k \in \mathcal{K} \tag{3e}$$
$$D_{k,j} \geq \sqrt{(x_j - x_k)^2 + (y_j - y_k)^2} \qquad \forall\ k \in \mathcal{K}, j \in \mathcal{K} \tag{3f}$$
$$\sum_k f_{i,k} \leq a_i \qquad \forall\ i \in \mathcal{I} \tag{3g}$$
$$\sum_i f_{i,k} \cdot cv_k = f_k \qquad \forall\ k \in \mathcal{K} \tag{3h}$$
$$f_k = \sum_j f_{k,j} \qquad \forall\ k \in \mathcal{K} \tag{3i}$$
$$\sum_k f_{k,j} = d_j \qquad \forall\ j \in \mathcal{J} \tag{3j}$$
$$w_k \geq z_{i,k} \qquad \forall\ i \in \mathcal{I}, k \in \mathcal{K} \tag{3k}$$
$$\sum_i z_{i,k} \geq w_k \qquad \forall\ k \in \mathcal{K} \tag{3l}$$
$$w_k \geq z_{k,j} \qquad \forall\ k \in \mathcal{K}, j \in \mathcal{J} \tag{3m}$$
$$\sum_j z_{k,j} \geq w_k \qquad \forall\ k \in \mathcal{K} \tag{3n}$$
$$0 \leq f_k \leq mc_k \cdot w_k \qquad \forall\ k \in \mathcal{K} \tag{3o}$$
$$0 \leq x_k \leq x_k^{\mathrm{U}} \cdot w_k \qquad \forall\ k \in \mathcal{K} \tag{3p}$$
$$0 \leq y_k \leq y_k^{\mathrm{U}} \cdot w_k \qquad \forall\ k \in \mathcal{K} \tag{3q}$$
$$0 \leq f_{i,k} \leq f_{i,k}^{\mathrm{U}} \cdot z_{i,k} \qquad \forall\ i \in \mathcal{I}, k \in \mathcal{K} \tag{3r}$$
$$D_{i,k}^{\mathrm{L}} \cdot z_{i,k} \leq D_{i,k} \leq D_{i,k}^{\mathrm{U}} \cdot z_{i,k} \qquad \forall\ i \in \mathcal{I}, k \in \mathcal{K} \tag{3s}$$
$$0 \leq f_{k,j} \leq f_{k,j}^{\mathrm{U}} \cdot z_{k,j} \qquad \forall\ k \in \mathcal{K}, j \in \mathcal{J} \tag{3t}$$
$$D_{k,j}^{\mathrm{L}} \cdot z_{k,j} \leq D_{k,j} \leq D_{k,j}^{\mathrm{U}} \cdot z_{k,j} \qquad \forall\ k \in \mathcal{K}, j \in \mathcal{J} \tag{3u}$$
$$w_k, z_{i,k}, z_{k,j} \in \{0, 1\} \qquad \forall\ i \in \mathcal{I}, k \in \mathcal{K}, j \in \mathcal{J} \tag{3v}$$

We assume that the facilities of the same type have the same costs and characteristics associated with them, i.e., ff_k, vf_k, mc_k, $ft_{i,k}$, $vt_{i,k}$, $ft_{k,j}$, $vt_{k,j}$ are the same $\forall\ k \in \mathcal{K}_n$. Therefore, in order to break the symmetry, we add Equations (3w)-(3w) to the formulation. These contraints enforce that for facilities k of the same type, i.e., $k \in \mathcal{K}_n,\ n \in \mathcal{N}$, the model will chose first to build the ones with the lower indexes, and those will be located in lower x_k coordinate.

$$w_k \geq w_{k+1} \qquad \forall\ k \in \mathcal{K}_n,\ n \in \mathcal{N} \tag{3w}$$
$$x_k \geq x_{k+1} \qquad \forall\ k \in \mathcal{K}_n,\ n \in \mathcal{N} \tag{3x}$$

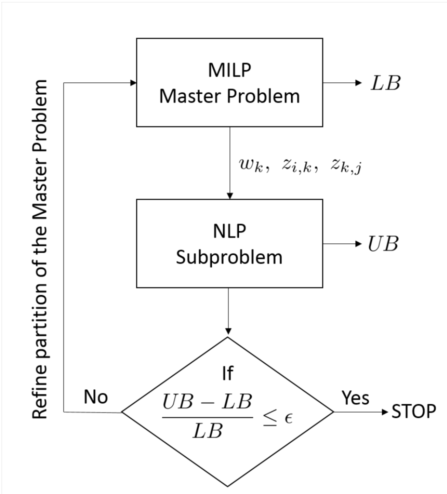

Fig. 2 Representation bilevel decomposition algorithm

3 Bilevel decomposition algorithm

Although global optimization solvers perform reasonably well for small-scale instances of the nonconvex MINLP problem (3), their performance scales poorly and it becomes computationally very expensive for mid to large-scale problems. For this reason, we propose a bilevel decomposition algorithm that consists of decomposing the problem into a master problem and a subproblem. The master problem is based on a relaxation of the nonconvex MINLP (3), which yields an MILP that predicts the selection of facilities and their links to suppliers and customers, as well as a lower bound on the cost of problem (2) or (3). The subproblem corresponds to a nonconvex NLP of reduced dimensionality that results from fixing the binary variables w_k, $z_{i,k}$, $z_{k,j}$ in the MINLP problem (3), according to the binary variables predicted in the MILP master problem. Figure 2 shows the algorithm.

3.1 Master problem

The non-linearity and nonconvexity of the formulation (3) arise from the fact that the distances are decision variables. If the coordinates for the potential facilities are fixed, the distances can be pre-computed and used as parameters in the model. In order to take advantage of this property, the master problem partitions the space into p sub-regions as represented in Figure 3.

In order to derive a valid relaxation for the original MINLP (3), we consider that each of the facilities can be located in each of the sub-regions. Therefore, it is possible to determine *a priori* the minimum distance between suppliers and facilities, and between facilities and customers. Specifically, by discretizing

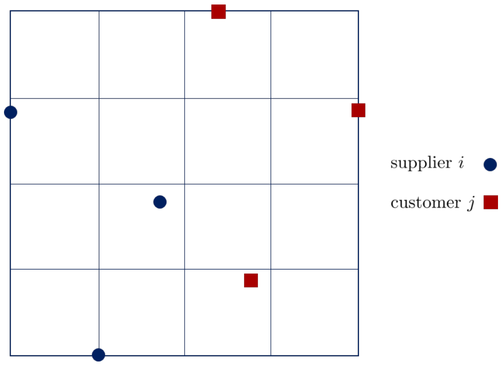

Fig. 3 Representation of the p sub-regions

the 2-dimensional space, we are able to pre-calculate the minimum distance between the fixed points and each sub-region p, $\widehat{D}_{i,p}$ and $\widehat{D}_{j,p}$, as follows:

$$dx_{i,p} = \max\{|x_i - x_p| - \bar{x}/2, 0\} \qquad \forall\ i \in \mathcal{I}, p \in \mathcal{P} \tag{4a}$$
$$dy_{i,p} = \max\{|y_i - y_p| - \bar{y}/2, 0\} \qquad \forall\ i \in \mathcal{I}, p \in \mathcal{P} \tag{4b}$$
$$dx_{j,p} = \max\{|x_j - x_p| - \bar{x}/2, 0\} \qquad \forall\ j \in \mathcal{J}, p \in \mathcal{P} \tag{4c}$$
$$dy_{j,p} = \max\{|y_j - y_p| - \bar{y}/2, 0\} \qquad \forall\ j \in \mathcal{J}, p \in \mathcal{P} \tag{4d}$$
$$\widehat{D}_{i,p} = \max\{\sqrt{dx_{i,p}^2 + dy_{i,p}^2}, D_{i,p}^{\mathrm{L}}\} \qquad \forall\ i \in \mathcal{I}, p \in \mathcal{P} \tag{4e}$$
$$\widehat{D}_{j,p} = \max\{\sqrt{dx_{j,p}^2 + dy_{j,p}^2}, D_{j,p}^{\mathrm{L}}\} \qquad \forall\ j \in \mathcal{J}, p \in \mathcal{P} \tag{4f}$$

where (x_p, y_p) are the coordinates of the mid-point of each sub-region p; $\bar{x}$ and $\bar{y}$ are the length of sub-region p in the x and y directions, respectively; $D_{i,p}^{\mathrm{L}}$ and $D_{i,p}^{\mathrm{L}}$ are the lower bounds for the distances, not allowing the model to chose to build a facility k on top of a fixed point from a supplier or a customer.

Making use of the partitions and the minimum distances, (4e)-(4f), the MINLP reformulation (3) can then be rewritten as an MILP in (5). The objective of this model is to decide which facilities k to build in each sub-region p and how to allocate the raw-material and products between suppliers i, customers j and these facilities k.

$$\min\ \Phi^{\mathrm{P}} = \sum_k \sum_p Cost_{k,p} + \sum_i \sum_k \sum_p Cost_{i,k,p} + \sum_k \sum_j \sum_p Cost_{k,j,p} \tag{5a}$$

$$Cost_{k,p} = ff_k \cdot w_{k,p} + vf_k \cdot f_{k,p} \qquad \forall\ k \in \mathcal{K}, p \in \mathcal{P} \tag{5b}$$
$$\begin{aligned} Cost_{i,k,p} = {} & cs_i \cdot f_{i,k,p} + ft_{i,k} \cdot z_{i,k,p} \\ & + vt_{i,k} \cdot \widehat{D}_{i,p} \cdot f_{i,k,p} \end{aligned} \qquad \forall\ i \in \mathcal{I}, k \in \mathcal{K}, p \in \mathcal{P} \tag{5c}$$
$$Cost_{k,j,p} = ft_{k,j} \cdot z_{k,j,p} + vt_{k,j} \cdot \widehat{D}_{j,p} \cdot f_{k,j,p} \qquad \forall\ k \in \mathcal{K}, j \in \mathcal{J}, p \in \mathcal{P} \tag{5d}$$

$$\sum_k \sum_p f_{i,k,p} \le a_i \qquad \forall\, i \in \mathcal{I} \tag{5e}$$

$$\sum_i f_{i,k,p} \cdot cv_k = f_{k,p} \qquad \forall\, k \in \mathcal{K}, p \in \mathcal{P} \tag{5f}$$

$$f_{k,p} = \sum_j f_{k,j,p} \qquad \forall\, k \in \mathcal{K}, p \in \mathcal{P} \tag{5g}$$

$$\sum_k \sum_p f_{k,j,p} = d_j \qquad \forall\, j \in \mathcal{J} \tag{5h}$$

$$w_{k,p} \ge z_{i,k,p} \qquad \forall\, i \in \mathcal{I}, k \in \mathcal{K}, p \in \mathcal{P} \tag{5i}$$

$$\sum_i z_{i,k,p} \ge w_{k,p} \qquad \forall\, k \in \mathcal{K}, p \in \mathcal{P} \tag{5j}$$

$$w_{k,p} \ge z_{k,j,p} \qquad \forall\, k \in \mathcal{K}, j \in \mathcal{J}, p \in \mathcal{P} \tag{5k}$$

$$\sum_j z_{k,j,p} \ge w_{k,p} \qquad \forall\, k \in \mathcal{K}, p \in \mathcal{P} \tag{5l}$$

$$\sum_p w_{k,p} \le 1 \qquad \forall k \in \mathcal{K} \tag{5m}$$

$$\sum_p z_{i,k,p} \le 1 \qquad \forall i \in \mathcal{I}, k \in \mathcal{K} \tag{5n}$$

$$\sum_p z_{k,j,p} \le 1 \qquad \forall k \in \mathcal{K}, j \in \mathcal{J} \tag{5o}$$

$$\sum_{p' \le p} w_{k',p'} \ge \sum_p w_{k,p} \qquad \forall\, k' < k,\ k \in \mathcal{K}_n,\ n \in \mathcal{N} \tag{5p}$$

$$0 \le f_{k,p} \le mc_k \cdot w_{k,p} \qquad \forall k \in \mathcal{K}, p \in \mathcal{P} \tag{5q}$$

$$0 \le f_{i,k,p} \le f^{\mathrm{U}}_{i,k,p} \cdot z_{i,k,p} \qquad \forall i \in \mathcal{I}, k \in \mathcal{K}, p \in \mathcal{P} \tag{5r}$$

$$0 \le f_{k,j,p} \le f^{\mathrm{U}}_{k,j,p} \cdot z_{k,j,p} \qquad \forall k \in \mathcal{K}, j \in \mathcal{J}, p \in \mathcal{P} \tag{5s}$$

$$w_{k,p}, z_{i,k,p}, z_{k,j,p} \in \{0,1\} \qquad \forall\, i \in \mathcal{I}, k \in \mathcal{K}, j \in \mathcal{J}, p \in \mathcal{P} \tag{5t}$$

Note that the MILP (5) has a considerably larger number of variables and constraints than the MINLP (3) since they must be defined for each partition p. It can easily be shown that the MILP master problem yields a lower bound to the total cost.

Proposition 1 *The MILP master problem (5) yields a lower bound to the original MINLP problem (3).*

Proof First we note that although the variables are disaggregated by partitions, the inequalities in (5m), (5n), (5o) ensure that only one facility and one link in the original MINLP are selected. Second, since we consider the shortest distance between the fixed points and the sub-regions as in (4e)-(4f), the transportation costs are underestimated. Thus, the MILP yields a Lower Bound (LB) to the original problem.

3.2 Subproblem

The subproblem consists of solving (3) for the fixed decisions of which facilities k to build, $\widehat{w}_k$, and how to allocate their material supply $\widehat{z}_{i,k}$ and products $\widehat{z}_{k,j}$ as selected in the MILP (5). The subproblem (6) is a reduced nonconvex NLP that is solved using a global optimization solver.

$$\begin{aligned} \min\ \Phi^{\mathrm{N}} &= \sum_k Cost_k + \sum_i \sum_k Cost_{i,k} + \sum_k \sum_j Cost_{k,j} \\ Cost_k &= ff_k \cdot \widehat{w}_k + vf_k \cdot f_k \qquad \forall\ k \in \mathcal{K} \end{aligned} \tag{6a}$$

$$Cost_{i,k} = cs_i \cdot f_{i,k} + ft_{i,k} \cdot \widehat{z}_{i,k} + vt_{i,k} \cdot f_{i,k} \cdot D_{i,k} \qquad \forall\ i \in \mathcal{I}, k \in \mathcal{K} \tag{6b}$$

$$Cost_{k,j} = ft_{k,j} \cdot \widehat{z}_{k,j} + vt_{k,j} \cdot f_{k,j} \cdot D_{k,j} \qquad \forall\ k \in \mathcal{K}, j \in \mathcal{J} \tag{6c}$$

$$D_{i,k} \geq \sqrt{(x_i - x_k)^2 + (y_i - y_k)^2} \qquad \forall\ i \in \mathcal{I}, k \in \mathcal{K} \tag{6d}$$

$$D_{k,j} \geq \sqrt{(x_j - x_k)^2 + (y_j - y_k)^2} \qquad \forall\ k \in \mathcal{K}, j \in \mathcal{J} \tag{6e}$$

$$\sum_k f_{i,k} \leq a_i \qquad \forall\ i \in \mathcal{I} \tag{6f}$$

$$\sum_i f_{i,k} \cdot cv_k = f_k \qquad \forall\ k \in \mathcal{K} \tag{6g}$$

$$f_k = \sum_j f_{k,j} \qquad \forall\ k \in \mathcal{K} \tag{6h}$$

$$\sum_k f_{k,j} = d_j \qquad \forall\ j \in \mathcal{J} \tag{6i}$$

$$0 \leq f_k \leq mc_k \cdot \widehat{w}_k \qquad \forall\ k \in \mathcal{K} \tag{6j}$$

$$0 \leq x_k \leq \bar{x}_k^{\mathrm{U}} \cdot \widehat{w}_k \qquad \forall\ k \in \mathcal{K} \tag{6k}$$

$$0 \leq y_k \leq \bar{y}_k^{\mathrm{U}} \cdot \widehat{w}_k \qquad \forall\ k \in \mathcal{K} \tag{6l}$$

$$0 \leq f_{i,k} \leq f_{i,k}^{\mathrm{U}} \cdot \widehat{z}_{i,k} \qquad \forall\ i \in \mathcal{I}, k \in \mathcal{K} \tag{6m}$$

$$\underline{D}_{i,k}^{\mathrm{L}} \cdot z_{i,k} \leq D_{i,k} \leq \bar{D}_{i,k}^{\mathrm{U}} \cdot \widehat{z}_{i,k} \qquad \forall\ i \in \mathcal{I}, k \in \mathcal{K} \tag{6n}$$

$$0 \leq f_{k,j} \leq f_{k,j}^{\mathrm{U}} \cdot \widehat{z}_{k,j} \qquad \forall\ k \in \mathcal{K}, j \in \mathcal{J} \tag{6o}$$

$$\underline{D}_{k,j}^{\mathrm{L}} \cdot \widehat{z}_{k,j} \leq D_{k,j} \leq \bar{D}_{k,j}^{\mathrm{U}} \cdot \widehat{z}_{k,j} \qquad \forall\ k \in \mathcal{K}, j \in \mathcal{J} \tag{6p}$$

The NLP subproblem (6), which comprises the original problem (3) for a fixed set of discrete decisions, yields an Upper Bound (UB). The key point in the NLP is that we update the bounds of the facilities coordinates such that their location (x_k, y_k) has to be within the bounds of the sub-region p chosen in (5), $\underline{x}_k^{\mathrm{L}} \leq x_k, \leq \bar{x}_k^{\mathrm{U}}$ and $\underline{y}_k^{\mathrm{L}} \leq y_k, \leq \bar{y}_k^{\mathrm{U}}$. This assumption greatly impacts on the tractability because the bounds for $D_{i,k}$ and $D_{k,j}$, which are part of the bilinear terms, become tighter ($\underline{D}^{\mathrm{L}}_{i,k} \leq D_{i,k} \leq \bar{D}^{\mathrm{U}}_{i,k}$ and $\underline{D}^{\mathrm{L}}_{k,j} \leq D_{k,j} \leq \bar{D}^{\mathrm{U}}_{i,k}$), and thus the McCormick convex envelopes [15] also become tighter, strengthening the lower bounds in the global optimization search of this NLP.

3.3 Algorithm

As discussed earlier in this section, the bilevel decomposition algorithm consists of iteratively solving the MILP master problem and the NLP subproblem, and refining the partitioning of space at each iteration. By shrinking the sub-regions, the minimum distances get closer to the actual distances, thus the lower bound becomes tighter; the selection of which facilities to build in each region gets closer to the optimal; and the bounds for $D_{i,k}$, and $D_{k,j}$ in the subproblem becomes tighter; hence, it is easier for the global optimization solver to find the optimal solution. The only drawback is that the size of the MILP master problem becomes larger, and thus harder to solve. The proposed procedure is shown in Algorithm 1.

Algorithm 1 Bilevel decomposition algorithm for CMWP

for a pre-specified optimality tolerance of ϵ **do**
  Determine the tightest rectangle $x \times y$ that includes all the fixed points;
  Partition this rectangle in $p_x \times p_y$;
  Set iteration $iter = 1$;
  while $gap > \epsilon$ **do**
    Solve the MILP master problem (5) and compute Lower Bound (LB^{iter});
    Fix the decision of which facilities k to build, w_k;
    Fix the decisions of how to allocate the materials, $z_{i,k}$, and products, $z_{k,j}$
    Fix the bounds for x_k and y_k according to the sub-region p that was selected for facility k to be built
    Solve the nonconvex NLP subproblem (6) using a global optimization solver, and compute an Upper Bound (UB^{iter})
    $gap = UB^{iter} - LB^{iter}$
    $p_x = p_x + N_x$ and $p_y = p_y + N_y$, where $N_x, N_y \in \mathbb{N}$
    $iter = iter + 1$
  end while
end for

The partitioning does not need to be uniform. For a fixed number of facilities, an nonuniform grid would perform better since only the sub-regions that had potential to place a facility would have their grid refined. Thus, the algorithm would not waste computational power in refining the sub-regions that will not have any facility. However, for the problem that is proposed in (2), the number of facilities k to be built is a decision variable. Based on experiments, the model tends to pick fewer and larger facilities in the early stages of the partitioning, and the structure of the network can completely change from one iteration to another. Therefore, we decided to adopt a uniform partitioning in this paper.

A bottleneck in the performance of this procedure is the solution of the subproblem, which is nonconvex NLP. However, it does not have to be solved to global optimality to yield a valid upper bound. Therefore, we specify a maximum solution time for the subproblem so that the algorithm does not waste time trying to achieve global optimality in the early iterations, when the bounds for $D_{i,k}$, and $D_{k,j}$ are still loose.

The proposed bilevel decomposition algorithm 1 converges to the global optimum in a finite number of steps with an ϵ-tolerance. We first establish the following proposition.

Proposition 2 *For an infinite number of partitions the MILP (5) and the MINLP (3) yield the same optimal solution Φ^*.*

Proof It trivially follows that for an infinite number of partitions the MILP (5) becomes an exact infinite dimensional representation of the MINLP (3). Thus, both (3) and (5) have the same optimal solution Φ^*.

Theorem 1 *Algorithm 1 converges in a finite number of iterations to the global optimum of problem (3) within an ϵ-tolerance at the bounds, $LB \leq UB - \epsilon$.*

Proof Algorithm 1 consists at solving a sequence of MILP problems (5) by increasing the number of partitions such that the set of partitions $\mathcal{P}^{iter}$ at iteration $iter$ is contained in the next iteration $\mathcal{P}^{iter} \subset \mathcal{P}^{iter+1}$. Thus, it follows that the lower bounds from (5) satisfy $\Phi^{iter} \leq \Phi^{iter+1}$. From Proposition 2 we have that $\Phi^{\infty} = \Phi^*$. Since we only consider a finite number of iterations, it follows that $\Phi^{iter} < \Phi^*$. Since the NLP subproblem (6) is solved to global optimality, it follows that $\Phi^{iter} < \Phi^* \leq \Phi^{\widehat{N}}$, where $\Phi^{\widehat{N}}$ is the incumbent, i.e. the best feasible solution of the NLP subproblem (6). For a given tolerance ϵ, a finite number of iterations $\widehat{iter}$ can be selected such that $\Phi^{\widehat{iter}} \leq \Phi^{\widehat{N}} - \epsilon$. Since $\Phi^{\widehat{iter}} = LB$, and $\Phi^{\widehat{N}} = UB$, $LB \leq UB - \epsilon$. Thus, the algorithm converges with an ϵ tolerance in a finite number of steps.

4 Small Test Problem

In order to test the Algorithm 1, we applied it first to a small test problem with 2 suppliers and 2 customer points. Supplier 1 is located at coordinates (0,0), and supplier 2 is located at (0,5). Customer 1 is located at coordinates (5,0), and customer 2 is located at (5,5). The fixed points of the network are represented in Figure 4. The cost of supply material from supplier 1 is 20 and from supplier 2 is 22. Both suppliers have an availability of 120, and both markets have a demand of 100.

There are 2 types of facilities. Type 1 has two potential facilities with a maximum capacity of 125 each, fixed cost of 7.18, and variable cost of 0.087. Type 2 has one potential facility with a maximum capacity of 250, fixed cost of

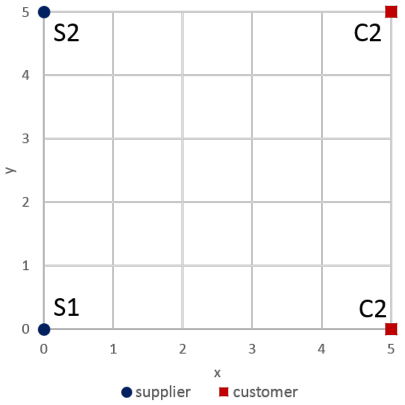

Fig. 4 Small test problem's network

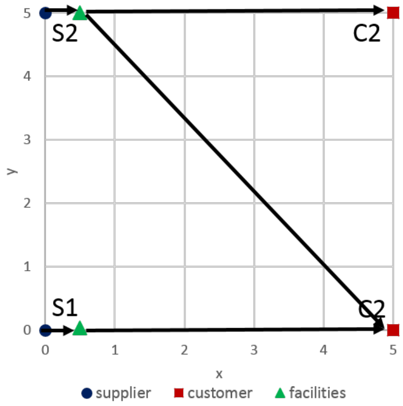

Fig. 5 Small test problem's optimal network

Table 1 Small test problem results

	Lower Bound	Upper Bound	Gap
$iter = 1$	4776.392	5159.830	8.028%
$iter = 2$	4916.468	5039.304	2.498%
$iter = 3$	4946.704	5039.304	1.872%
$iter = 4$	4968.799	5039.304	1.419%
$iter = 5$	4982.116	5039.304	1.148%
$iter = 6$	4991.011	5039.304	0.968%
$iter = 7$	4997.371	5039.304	0.839%
$iter = 8$	5002.145	5039.304	0.743%
$iter = 9$	5005.859	5039.304	0.668%
$iter = 10$	5008.832	5039.304	0.608%
$iter = 11$	5011.265	5039.304	0.560%
$iter = 12$	5013.293	5039.304	0.519%
$iter = 13$	5015.009	5039.304	0.484%

10.77, and variable cost of 0.067. All the facilities have a conversion to product flow of 90%.

The fixed transportation costs, $ft_{i,k}$ and $ft_{i,k}$, are 10, and the variable transportation costs, $vt_{i,k}$ and $vt_{i,k}$, are 0.3 for any type of link. It is assumed that the minimum allowed distance between a fixed supply or customer point and a facility is 0.5. The MILP master problem is solved to 0.01% optimality gap, and the maximum CPU time allowed for the NLP subproblem is 200 seconds. The monolithic MINLP version of the problem, (3), has 68 constraints, 15 binary variables, and 38 continuous variables.

Starting from p_x and p_y equal 1, i.e., no partition, and increasing them by 1 at each iteration, it takes 13 iterations and 11.87 seconds to solve this problem to 0.5% optimality tolerance. The lower bound, upper bound and optimality gap at each iteration are reported in Table 1.

As one can see, the lower bound gradually tightens up as the number of iterations $iter$, and consequently the number of partitions increase. The optimal solution of 5039.304 is the same as found by the general purpose global optimization solvers. However, while the algorithm solved this problem within an optimality tolerance of 0.5% in 11.87 seconds, BARON solved it in 1247.10 seconds, and ANTIGONE and SCIP could not solve it in 1 hour. The optimal network is shown in Figure 5.

5 Computational Results

In order to compare the performance of our proposed algorithm with currently available general purpose global optimization solvers, we randomly generated 12 test cases. The network varies in size as follows.

- Network 1: 2 suppliers × 2 consumers;
- Network 2: 5 suppliers × 5 consumers;
- Network 3: 10 suppliers × 10 consumers;
- Network 4: 20 suppliers × 20 consumers;
- Network 5: 40 suppliers × 40 consumers;

The 5 network structures are represented in Figures 6-10.

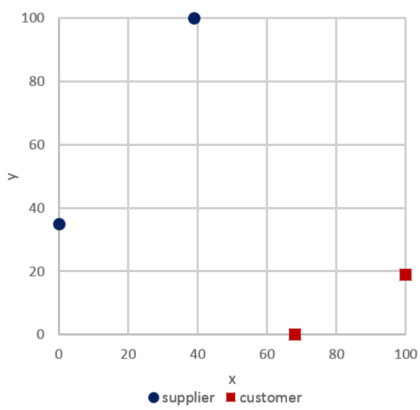

Fig. 6 Network 1

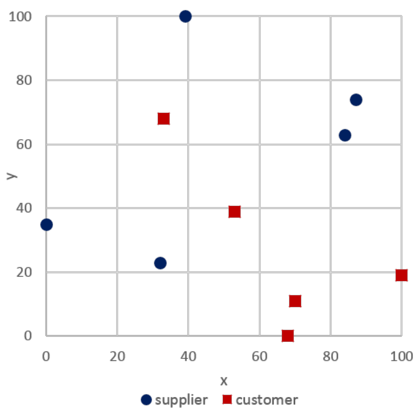

Fig. 7 Network 2

For each of the network options, the choice of 1, 2 and 3 types of facilities were tested, such that:

- Type 1: up to 2 large-scale facilities
- Type 2: up to 10 mid-scale facilities;
- Type 3: up to 20 small-scale facilities;

Therefore, for each of the network structure the problem was solved for 2, 12 and 32 potential facilities.

Each test case was solved using Algorithm 1 and by general purpose global optimization solvers, BARON, ANTIGONE and SCIP. We set the optimality

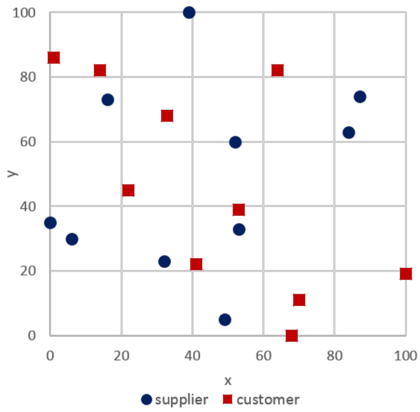

Fig. 8 Network 3

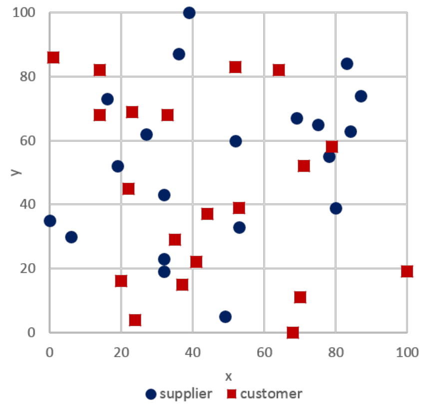

Fig. 9 Network 4

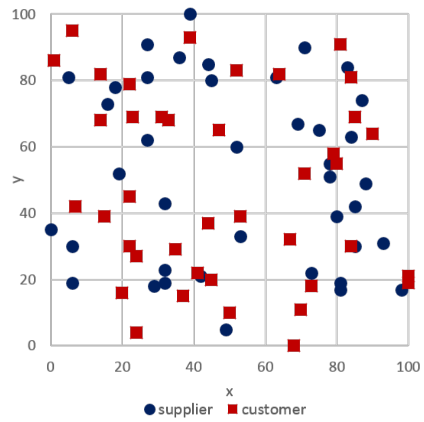

Fig. 10 Network 5

tolerance to 1% and the maximum total CPU time to 1 hour. Regarding the algorithm, it is required that the master problem has to be solved to 0.1% optimality gap, and it is allowed a maximum CPU time of 200 seconds for the solution of each NLP subproblem. We start the algorithm with a 10×10 partitioning of the space and at each iteration this partitioning is increased by $N_x, N_y = 5$.

Our computational tests were performed on a standard desktop computer with an Intel(R) Core(TM) i7-2600 CPU @ 3.40 GHz processor, with 8GB of RAM, running on Windows 7. We implemented the monolithic formulation and the global optimization algorithm in GAMS 24.7.1, solve the MILPs using CPLEX version 12.6.3 [11], the NLPs using BARON version 16.3.4 [21], and the MINLPs using BARON version 16.3.4 [21], ANTIGONE [16], and SCIP [10].

Table 2 Monolithic MINLP formulation size

	Binary Variables	Continuous Variables	Constraints
N1-T1	10	27	49
N2-T1	22	51	91
N3-T1	42	91	161
N4-T1	82	171	301
N5-T1	162	331	581
N1-T2	197	137	329
N2-T2	132	281	551
N3-T2	252	521	921
N4-T2	492	1001	1661
N5-T2	972	1961	3141
N1-T3	160	357	1089
N2-T3	352	741	1671
N3-T3	672	1381	2641
N4-T3	1312	2661	4581
N5-T3	2592	5221	8461

The case-studies are named such that the first 2 letters represent the network (i.e., N1, N2, N3, N4, and N5, represent Network 1, 2, 3, 4, and 5, respectively), and the last 2 letters represent the number of facility types considered (i.e., T1, T2, T3 represent 1 type, 2 types and 3 types, respectively). The size of monolithic MINLP formulation (3) for each of the test cases is shown in Table 2, and their results are shown in Table 3.

The results in Table 3 show that the bilevel decomposition algorithm was able to find the optimal solution within 1% optimality tolerance in 87% of the the case studies, and performed better than the other general purpose optimization solvers in 73% of them. It can be noticed that the improvement in performance due to the use of the algorithm is clearer for larger instances, specifically the networks with higher number of supplier and customer fixed points. The global optimization solver that performed the best for this type of problem was BARON. Antigone was the global solver that had the worst performance, not being able to find a feasible solution in 47% of the test cases. The performance curves for the Algorithm 1 and each of the global solvers are shown in Figure 11 [9].

6 Conclusions

In this paper we have presented a new version of the Capacitated Multi-facility Weber Problem that has fixed costs, multiple types of facilities, and two sets of fixed points representing suppliers and consumers. We have proposed a GDP formulation for this problem, which was reformulated as an MINLP, and then introduced a bilevel decomposition algorithm for this nonconvex problem. We prove that this algorithm converges to the global optimal within an ϵ tolerance in a finite number of iterations.

Table 3 Computational experiments' results

		Algorithm 1	BARON	ANTIGONE	SCIP
N1-T1	Minimum Cost	10.537	10.537	10.537	10.537
	Gap	0.4%	1.0%	1.0%	1.0%
	CPU Time (s)	0.725	0.090	0.267	0.360
N2-T1	Minimum Cost	23.850	23.850	23.850	23.850
	Gap	0.8%	1.0%	1.0%	1.0%
	CPU Time (s)	5.046	2.480	8.014	12.780
N3-T1	Minimum Cost	44.913	44.957	44.913	44.932
	Gap	0.9%	1.0%	1.0%	3.3%
	CPU Time (s)	45.854	164.330	225.297	3600
N4-T1	Minimum Cost	60.410	60.416	60.617	60.491
	Gap	0.8%	1.6%	1.0%	4.18%
	CPU Time (s)	340.541	3600	3183.016	3600
N5-T1	Minimum Cost	91.926	92.515	Infeasible	93.253
	Gap	0.8%	3.5%	NA	8.4%
	CPU Time (s)	705.269	3600	0.311	3600
N1-T2	Minimum Cost	10.537	10.537	10.537	10.537
	Gap	0.4%	1.0%	1.0%	1.0%
	CPU Time (s)	3.243	1.910	1.093	2.040
N2-T2	Minimum Cost	23.850	23.850	23.850	23.850
	Gap	0.8%	1.0%	1.0%	1.0%
	CPU Time (s)	12.055	18.810	280.717	73.980
N3-T2	Minimum Cost	44.913	44.913	No solution returned	44.989
	Gap	0.9%	1.0%	NA	3.6%
	CPU Time (s)	50.038	3114.320	3600	3600
N4-T2	Minimum Cost	60.411	60.425	No solution returned	61.042
	Gap	0.8%	2.0%	NA	5.73%
	CPU Time (s)	627.567	3600	3600	3600
N5-T2	Minimum Cost	91.966	92.466	Infeasible	94.362
	Gap	1.2%	5.5%	NA	10.2%
	CPU Time (s)	3600	3600	7.491	3600
N1-T3	Minimum Cost	3.921	3.921	3.921	3.921
	Gap	0.6%	1.0%	1.0%	1.0%
	CPU Time (s)	15.520	3.921	47.262	3.480
N2-T3	Minimum Cost	23.850	23.850	23.850	23.850
	Gap	0.8%	1.0%	1.0%	1.0%
	CPU Time (s)	51.521	93.350	726.475	118.430
N3-T3	Minimum Cost	44.932	44.913	No solution returned	44.989
	Gap	0.9%	1.0%	NA	3.46%
	CPU Time (s)	80.239	1697.660	3600	3600
N4-T3	Minimum Cost	60.408	60.459	No solution returned	61.235
	Gap	0.8%	2.1%	NA	5.7%
	CPU Time (s)	2461.583	3600	3600	3600
N5-T3	Minimum Cost	92.621	92.2545	Infeasible	94.2516
	Gap	2.0%	5.4%	NA	10.1%
	CPU Time (s)	3600	3600	57.453	3600

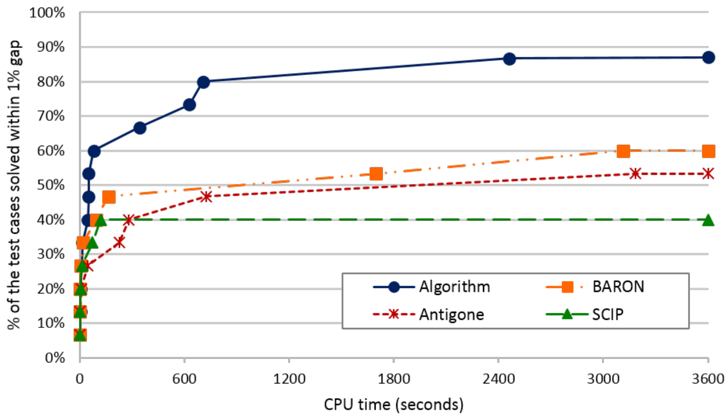

Fig. 11 Performance curves comparing the bilevel decomposition algorithm with global optimization solvers

We test the algorithm for 15 test cases varying from 2 suppliers and 2 consumers, to 40 suppliers and 40 consumers, from 1 to 3 types of facilities, and from 2 to 32 potential facilities, and compare the results with general purpose global optimization solvers. The results show that our algorithm performs more efficiently for 73% of the test cases within 1% of optimality gap, and that the improvement in performance is more noticeable for larger instances.

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