MILP reformulation of the multi-echelon stochastic inventory system with uncertain demands

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1 Introduction

In this article we present an effective mixed-integer linear programming (MILP) formulation for design of a multi-echelon stochastic inventory system with uncertain customer demands. In You and Grossmann [2010] a three-echelon supply chain with inventories under uncertainty is presented. In that supply chain, the location of the plants and the customer demand zones (CDZ) are known. Potential distribution centers (DC) are given and the objective is to decide which DCs to install in order to minimize total costs, which include transportation cost, installation cost as well as inventory holding costs. The formulation also determines the service times for each DC, and what the size of the safety stock should be at all DCs and CDZs. This model uses single sourcing, which is often the case for supply chains in industrial gases or specialty chemicals. That means that all DCs are served by only one plant and each CDZ is served by only one DC. The detailed model can be found in You and Grossmann [2010].

In this short note we first reformulate the mixed-integer nonlinear programming (MINLP) model presented in You and Grossmann [2010] and You and Grossmann [2011], using an alternative linearization scheme leading to a model that is significantly smaller in size, and tighter. We also present a successive piecewise linear approximation with which we can solve the model with a sufficiently small optimality gap without the need of a global optimization solver.

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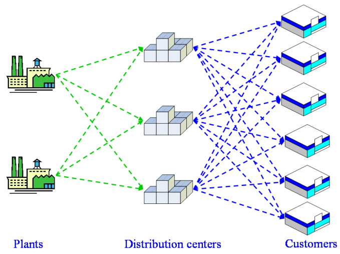

Figure 1: Network of multiple plants, DCs and CDZs

2 Model Formulation

In You and Grossmann [2010] the original MINLP model (P0) is stated as:

$$Min : \sum_{j\in J} f_j Y_j + \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} A_{ijk} X_{ij} Z_{jk} + \sum_{j\in J}\sum_{k\in K} B_{jk} Z_{jk} \tag{1}$$

$$+ \sum_{j\in J} q1_j \sqrt{N_j \sum_{k\in K} \sigma_k^2 Z_{jk}} + \sum_{k\in K} q2_k \sqrt{L_k} \tag{2}$$

s.t.

$$N_j \geq \sum_{i\in I} (SI_i + t1_{ij}) X_{ij} - S_j, \quad \forall j, \tag{3}$$

$$L_k \geq \sum_{j\in J} (S_j + t2_{jk}) Z_{jk} - R_k, \quad \forall k, \tag{4}$$

$$\sum_{i\in I} X_{ij} = Y_j, \quad \forall j, \tag{5}$$

$$\sum_{j\in J} Z_{jk} = 1, \quad \forall k, \tag{6}$$

$$Z_{jk} \leq Y_j, \quad \forall j,k, \tag{7}$$

$$X_{ij}, Y_j, Z_{jk} \in \{0,1\}, \quad \forall i,j,k, \tag{8}$$

$$S_j \geq 0, N_j \geq 0, \quad \forall j, \tag{9}$$

$$L_k \geq 0, \quad \forall k, \tag{10}$$

with the following parameters(see section 9 for complete nomenclature):

$$A_{ijk} = (c1_{ij}\chi + \theta 1_j t1_{ij}) \cdot \mu_k,$$

$$B_{jk} = (g_j\chi + c2_{jk}\chi + \theta 2_k t2_{jk}) \cdot \mu_k,$$

$$q1_j = \lambda 1_j \cdot h1_j,$$

$$q2_k = \lambda 2_k \cdot h2_k \cdot \sigma_k.$$

In order to decrease the number of nonlinear terms, the authors linearize in a similar way as in Glover [1975], the bilinear terms in (P0) to obtain a new formulation, (P1), with fewer nonlinear constraints. In the objective function, Eq. (2), the product of two binaries $X_{ij}Z_{jk}$ is linearized by introducing continuous variables XZ_{ijk} and the following constraints:

$$XZ_{ijk} \leq X_{ij}, \quad \forall i, j, k, \tag{11}$$

$$XZ_{ijk} \leq Z_{jk}, \quad \forall i, j, k, \tag{12}$$

$$XZ_{ijk} \geq X_{ij} + Z_{jk} - 1, \quad \forall i, j, k, \tag{13}$$

$$XZ_{ijk} \geq 0, \quad \forall i, j, k, \tag{14}$$

Since the objective is to minimize the cost, Eqs. (11) and (12) can be removed without affecting the solution. In Eq. (4) the bilinear terms consisting of a continuous variable (S_j) times a binary variable Z_{jk} are linearized by, introducing two new continuous variables, SZ_{jk} and $SZ1_{jk}$, as well as the following constraints:

$$SZ_{jk} + SZ1_{jk} = S_j, \quad \forall j, k, \tag{15}$$

$$SZ_{jk} \leq Z_{jk} \cdot S_j^U, \quad \forall j, k, \tag{16}$$

$$SZ1_{jk} \leq (1 - Z_{jk}) \cdot S_j^U, \quad \forall j, k, \tag{17}$$

$$SZ_{jk} \geq 0, SZ1_{jk} \geq 0 \quad \forall j, k. \tag{18}$$

In a similar way, the bilinear terms between the continuous variables N_j and the binary variables Z_{jk}, in the objective function, are linearized as follows:

$$NZ_{jk} + NZ1_{jk} = N_j, \quad \forall j, k, \tag{19}$$

$$NZ_{jk} \leq Z_{jk} \cdot N_j^U, \quad \forall j, k, \tag{20}$$

$$NZ1_{jk} \leq (1 - Z_{jk}) \cdot N_j^U, \quad \forall j, k, \tag{21}$$

$$NZ_{jk} \geq 0, NZ1_{jk} \geq 0 \quad \forall j, k. \tag{22}$$

And finally, the product under the square root term in the objective function is replaced with a variable NZV_j:

$$NZV_j = \sum_{k \in K} \sigma_k^2 \cdot NZ_{jk}, \quad \forall j \tag{23}$$

Although these linearization schemes are correct, they can be replaced with more effective formulations as will be shown next.

3 Alternative linearizations

In this section we show how to linearize the model P0 into an reformulated model R1 which is more compact than the reformulated model P1. First, the bilinear terms, $\sum_{i \in I} \sum_{j \in J} \sum_{k \in K} A_{ijk} X_{ij} Z_{jk}$, in the objective function are similar to the objective in the quadratic assignment problem presented by Koopmans and Beckmann [1957]. Therefore, we can use a similar approach, as in [Nyberg and Westerlund, 2012], to rearrange the variables in the objective function. Since the parameter μ_k in $A_{ijk} = (c1_{ij}\chi + \theta 1_j t1_{ij}) \cdot \mu_k$ is only dependent on the index k we can rewrite the objective function as follows:

$$\sum_{j \in J} (\sum_{i \in I} A_{ij} X_{ij}) \sum_{k} Z_{jk} \mu_k, \tag{24}$$

where

$$A_{ij} = (c1_{ij}\chi + \theta 1_j t1_{ij}), \tag{25}$$

is a new constant independent of k. According to Eq. (5) at most one of the X_{ij} variables in $\sum_i A_{ij}X_{ij}$ can be nonzero. Thus, this term can now be linearized using $(|I|+1)\cdot j$ new continuous variables, XZ_{ij}, instead of $2\cdot|I|\cdot|J|\cdot|K|$ as in Eq. (2). Hence, for the linearization, the objective function is written as:

$$\sum_{j\in J}\sum_{i\in I} A_{ij}XZ_{ij}, \tag{26}$$

and the new constraints (instead of Eqs. (11) to (14)) as,

$$XZ_{0j} + \sum_i XZ_{ij} = \sum_k \mu_k Z_{jk} \quad \forall j, \tag{27}$$

$$XZ_{ij} \le \sum_k \mu_k X_{ij} \quad \forall i,j, \tag{28}$$

$$XZ_{0j} \le \sum_k \mu_k(1-Y_j) \quad \forall j, \tag{29}$$

where the last variable Y_j comes from Eq. (5).

Since the objective is to minimize, and all variables are nonnegative we can substitute the N_j in the objective function, with the right hand side ($\sum_{i\in I}(SI_i + t1_{ij})X_{ij} - S_j$) of Eq. (3). Instead of having bilinear terms between the continuous variables N_j and the binary variables Z_{jk}, we can now linearize the expression below as follows:

$$(\sum_{i\in I}(SI_i + t1_{ij})X_{ij} - S_j)\cdot(\sum_{k\in K} Z_{jk}\sigma_k^2), \tag{30}$$

which can be written as:

$$(\sum_{i\in I}(SI_i + t1_{ij})X_{ij})\cdot(\sum_{k\in K} Z_{jk}\sigma_k^2) - S_j\cdot(\sum_{k\in K} Z_{jk}\sigma_k^2). \tag{31}$$

Again, it can be noted that according to Eq. (5) at most one of the binary X_{ij} variables in the above equation can be nonzero. The first part of this term can therefore be linearized analogously as in Eqs. (27) to (29). Note that the variables are exactly the same in the bilinear terms, and therefore the same linearizations would work in both cases if it were not for the constants preceding the variables. However, we choose to write these linearizations with new continuous variables ZX_{ij} as well since this formulation is tighter than the one presented in You and Grossmann [2010].

$$ZX_{0j} + \sum_i ZX_{ij} = \sum_k \sigma_k^2 Z_{jk} \quad \forall j, \tag{32}$$

$$ZX_{ij} \le \sum_k \sigma_k^2 x_{ij} \quad \forall i,j, \tag{33}$$

$$ZX_{0j} \le \sum_k \sigma_k^2(1-Y_j) \quad \forall j, \tag{34}$$

In the second half of Eq. (31), the bilinear terms in $S_j \cdot (\sum_{k\in K} Z_{jk}\sigma_k^2)$ are again exactly the same as the ones already linearized in Eqs. (15) to (17). Therefore the bilinear terms are already defined earlier and we can write the expression under the first square root term as:

$$\sum_{i\in I} S1_{ij} \cdot ZX_{ij} - \sum_{k\in K} \sigma_k^2 \cdot SZ_{jk} \tag{35}$$

4 Reformulated model

Based on the linearizations in the previous section, the reformulated nonlinear model (R1) is as follows:

$$\begin{aligned} Min: &\sum_{j\in J} f_j Y_j + \sum_{j\in J}\sum_{i\in I} A_{ij} XZ_{ij} + \sum_{j\in J}\sum_{k\in K} B_{jk} Z_{jk} \\ &+ \sum_{j\in J} q1_j \sqrt{\sum_{i\in I} S1_{ij} \cdot ZX_{ij} - \sum_{k\in K} \sigma_k^2 \cdot SZ_{jk}} + \sum_{k\in K} q2_k \sqrt{L_k} \end{aligned} \tag{36}$$

s.t.

$$L_k \geq \sum_{j\in J} SZ_{jk} + t2_{jk} \cdot Z_{jk} - R_k, \quad \forall k, \tag{37}$$

$$\sum_{i\in I} X_{ij} = Y_j, \quad \forall j, \tag{38}$$

$$\sum_{j\in J} Z_{jk} = 1, \quad \forall k, \tag{39}$$

$$Z_{jk} \leq Y_j, \quad \forall j,k, \tag{40}$$

$$XZ_{0j} + \sum_i XZ_{ij} = \sum_k \mu_k Z_{jk} \quad \forall j, \tag{41}$$

$$XZ_{ij} \leq \sum_k \mu_k x_{ij} \quad \forall i,j, \tag{42}$$

$$XZ_{0j} \leq \sum_k \mu_k (1 - Y_j) \quad \forall j, \tag{43}$$

$$ZX_{0j} + \sum_i ZX_{ij} = \sum_k \sigma_k^2 Z_{jk} \quad \forall j, \tag{44}$$

$$ZX_{ij} \leq \sum_k \sigma_k^2 x_{ij} \quad \forall i,j, \tag{45}$$

$$ZX_{0j} \leq \sum_k \sigma_k^2 (1 - Y_j) \quad \forall j, \tag{46}$$

$$SZ_{jk} + SZ1_{jk} = S_j, \quad \forall j,k, \tag{47}$$

$$SZ1_{jk} \leq (1 - Z_{jk}) \cdot S_j^U, \quad \forall j,k, \tag{48}$$

where all varialbes are ≥ 0. As shown later on in Table 3 this formulation is significantly smaller in size as well as tighter than P1.

5 Concave nonlinear terms

In the same manner as in You and Grossmann [2010] the only remaining nonlinear terms in R1 can be underestimated to obtain an MILP formulation (R2) of the model, which provides a good starting solution as well as a lower bound for R1. Since we use the same variables and constraints, every feasible solution of the MILP R2 is also a feasible solution to the MINLP R1. Since we underestimate R1, a valid lower bound for R2 is also a lower bound for R1. In You and Grossmann [2010] (P1) is relaxed in order to obtain the MILP P2, which is solved with CPLEX to obtain a good starting solution for the MINLP. You and Grossmann [2010] underestimate the nonlinear square root terms $\sqrt{L_k}$ and $\sqrt{NZV_j}$ with their respective secants $\frac{L_k}{\sqrt{L_k^{UB}}}$ and $\frac{NZV_j}{\sqrt{NZV_j^{UB}}}$ as proposed by Soland [1971]. However, since one of the binary Z_{jk} variables in Eq. (4) is nonzero, the lower bound for L_k, when R_k is a parameter becomes:

$$L_k^{LB} = \min_{j \in J}\{t2_{jk} - R_k\} \quad \forall k. \tag{49}$$

In order to obtain a tighter underestimator, we use the following function, which is the secant in the feasible region of L_k:

$$\frac{\sqrt{L_k^{LB}} - \sqrt{L_k^{UB}})}{((L_k^{LB}) - L_k^{UB})} L_k + \sqrt{L_k^{UB}} - \frac{(\sqrt{L_k^{LB}} - \sqrt{L_k^{UB}})}{((L_k^{LB}) - L_k^{UB})} L_k^{UB} \tag{50}$$

In Fig. 2 the difference is shown between underestimating the square root with the provided lower bounds versus underestimating with the lower bound 0. This is not the case when responsiveness R_k is defined as a variable as in You and Grossmann [2011].

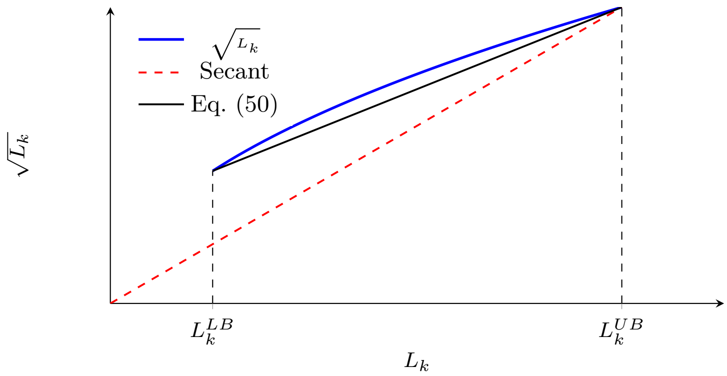

Figure 2: Underestimators with lower bounds and with lower bound set to zero

5.1 Piecewise linear approximation

In order to tighten the underestimations for the convex square root terms, we use a sequential piecewise linear approximation approach. Since the MILP R2 formulation gives a relatively small gap, even on larger instances, we introduce a successive piecewise linear approximation scheme in order to obtain sufficiently

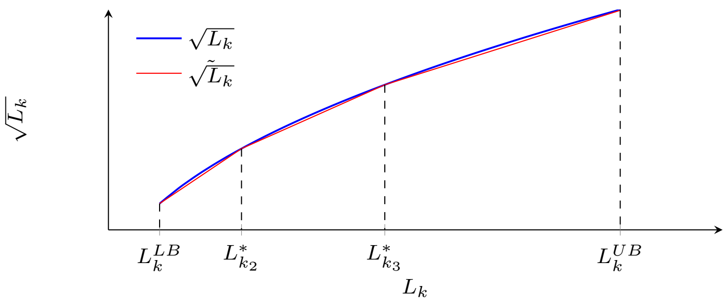

Figure 3: Intervals with the sequential approach after a few iterations

small gaps. We use the $\delta-$formulation presented in Dantzig [1963] and whose tightness has been studied by Padberg [2000]. Since there are $|J| + |K|$ square root terms in the model, it is not a good idea to add too many intervals per square root term. Therefore, we solve the underestimating MILP several times, and only partition the terms that are greater than zero in the underestimation. In the equations below, we show the piecewise linearization used for the term $\sqrt{L_k}$. In the model we also partition the second square root term analogously.

$$L_k = L_k^{LB} + \sum_{n=1}^{N} \delta_n, \tag{51}$$

$$\tilde{\sqrt{L_k}} = \sqrt{L_k^{LB}} + \sum_{n=1}^{N} \frac{\sqrt{L_{k_{n+1}}^*} - \sqrt{L_{k_n}^*}}{L_{k_{n+1}}^* - L_{k_n}^*} \delta_n, \tag{52}$$

$$(L_{k_{n+1}}^* - L_{k_{n+}}^*) w_n \leq \delta_n \leq (L_{k_{n+1}}^* - L_{k_n}^*) w_{n-1}, \quad n = 2, \ldots, N-1, \tag{53}$$

$$(L_{k_2}^* - L_{k_1}^*) w_1 \leq \delta_n \leq L_{k_2}^* - L_{k_1}^*, \tag{54}$$

$$\delta_n \leq (L_k^{UB} - L_{k_N}^*) w_{N-1}, \tag{55}$$

$$w_n \in \{0, 1\}, n = 1, \ldots, N-1. \tag{56}$$

In the sequential approach, we start by solving the MILP model R2, then we add a grid point for each square root term that is larger than its lower bound. The MILP will therefore be exact in the solution found. We then solve the model again with the new grid points and if we find a better solution we again add grid points at the new solution. Therefore, N, which is the number of grid points, will be different for all square root terms. Most of the terms will stay at zero, and therefore no new grid points or variables will be needed for those terms. After no improvement is found, the final MILP with the added grid points is solved to a predefined optimality gap. Since the piecewise underestimator is exact in the grid points, this approach can also be used to close the gap completely. This is a similar approach as the branch-and-refine method presented in Leyffer et al. [2008] and You et al. [2011].

6 Computational Results

In this section, we present computational results on the same size instances as in You and Grossmann [2010]. All computations in this paper are conducted on an Asus UX31E ultra book with a 2.8 GHz quad-core Intel processor and 4 Gb of ram. As a MILP solver we used CPLEX 12.3 with the default parameters. All constant values in the model are generated randomly with uniform distribution within predefined values. These values can be found in section 9. In order to be able to reproduce the runs, the default random seed in GAMS 23.7.3 was used. The difficulty of these problems are strongly dependent on the values assigned to the different costs.

			P2			R2		
$\|I\|$	$\|J\|$	$\|K\|$	Bin. Vars.	Con. Vars.	Const.	Bin. Vars.	Cont. Vars.	Const.
2	20	20	460	2,480	5,300	460	980	1,840
5	30	50	1,680	13,640	33,190	1,680	3470	6580
10	50	100	5,550	70,250	185,350	5,550	11,200	21,500
20	50	100	6,050	120,250	335,350	6,050	12,200	22,400
3	50	150	7,700	52,800	120,450	7,700	15,650	30,900
15	100	200	21,600	300,300	1,040,700	21,600	43,400	83,800

Table 1: Sizes of the old and reformulated models

In Table 1 the model sizes of P2 and R2 are compared. As can be seen, the difference is very large, especially for the larger instances. The root node values of the relaxed LP for the different formulations is shown in Table 2. As can be seen from Table 1 and Table 2, R2 is both smaller in size and at least as tight as P2.

			P2	R2
$\|I\|$	$\|J\|$	$\|K\|$	LB	LB
2	20	20	1,024,834	1,026,054
5	30	50	2,087,756	2,087,756
10	50	100	3,550,280	3,550,280
20	50	100	3,234,617	3,234,617
3	50	150	4,779,460	4,779,460
15	100	200	6,468,441	6,468,441

Table 2: Comparison of the tightness in the root node of the relaxed MILPs P2 and R2

In Table 3 the computational comparisons of the reformulated MILP (R2) and the original formulation (P2) are presented. As can be observed, P2 can only be solved for the smallest instance within the time limit of 10 minutes. On the other hand for the new formulation R2, all instances are solved in a few seconds. This comparison is made with the same secant function for the square root terms in both cases (i.e. zero lower bound in Eq. (50)). If the square root term in R2 is relaxed with the tighter underestimator shown in section 5, then the optimal values of the MILP will be higher and therefore closer to the optimal value of R1. However the solution times for R2 would still be as fast as in Table 3.

Table 4 shows the global optimum of the MILP R2, when underestimating the square root terms with the tighter lower bound in Eq. (50) and the corresponding calculated feasible solution of the problem when evaluating the square root terms. Note that the optimal solutions for R2 are higher in Table 4 than in Table 3 for the same problems, while the solution times are roughly the same.

$\|I\|$	$\|J\|$	$\|K\|$	P2 UB	P2 LB	P2 Gap	P2 Time(s)	R2 UB	R2 LB	R2 Gap	R2 Time(s)
2	20	20	1,584,557	1,584,557	0.00%	1.27	1,584,557	1,584,557	0.00%	0.09
5	30	50	3,248,076	2,988,151	7.80%	600	3,169,880	3,169,880	0.00%	0.58
10	50	100	6,893,324	3,569,929	48.20%	600	5,173,888	5,173,888	0.00%	4.18
20	50	100	5,969,190	3,245,514	45.63%	600	4,602,706	4,602,706	0.00%	6.15
3	50	150	10,852,632	5,009,423	53.84%	600	7,151,942	7,151,942	0.00%	2.92
15	100	200	54,755,897	6,407,108	88.30%	600	8,816,596	8,816,596	0.00%	23.4

Table 3: Comparison of the performance of the MILPs P2 and R2, with a time limit of 600 seconds

$\|I\|$	$\|J\|$	$\|K\|$	MILP (R2)	MINLP (R1)	Gap	Time(s)
2	20	20	1,708,930	1,783,540	4.18%	0.09
5	30	50	3,538,980	3,770,404	6.14%	0.67
10	50	100	5,862,307	6,215,596	5.68%	4.74
20	50	100	5,294,980	5,396,136	1.87%	6.86
3	50	150	8,161,606	8,479,354	3.75%	1.65
15	100	200	10,318,630	10,727,860	3.81%	28.2

Table 4: Gap between the optimal solution of R2 and the calculated feasible MINLP solution

In Table 5 solution times for the sequential piecewise approach presented in section 5.1 are shown. Even the larger instances can be solved within reasonable computational time. However, the solution times for different instances with the same size, can be completely different. This formulation is strongly dependent on the values of the parameters. This is also why the solution time is longer for the third instance in Table 5 than for the fourth, even though the fourth instance is larger in size.

$\|I\|$	$\|J\|$	$\|K\|$	Solution	LB	Gap	Time(s)
2	20	20	1,776,969	1,769,882	0.40%	0.94
5	30	50	3,728,020	3,704,300	0.64 %	249
10	50	100	6,182,142	6,120,321	0.99%	9,042
20	50	100	5,396,821	5,368,078	0.53%	1,042
3	50	150	8,489,428	8,409,329	0.94%	415
15	100	200	10,704,022	10,597,052	0.99%	10,840

Table 5: Solutions for different size instances with the sequential piecewise approach

7 Conclusions

In this short note, we have showed that by reformulating the three-stage multi-echelon inventory system with specific exact linearizations, we can solve larger problems directly with an MILP solver without the need of decomposing the problem. The new formulation is significantly smaller in size, both in the number of variables as well as in the number of constraints. An MILP underestimation of the problem can be solved very quickly in order to obtain a good feasible solution for the MINLP. The paper shows a simple sequential piecewise approximation scheme that can be used to solve the problem within a desired optimality gap.

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9 Notation

Models

P0, original MINLP
P1, original reformulated MINLP
P2, original underestimating MILP
R1, reformulated P1
R2, reformulated P2

Sets

I = set of suppliers
J = set of candidate DC locations
K = set of CDZs

Parameters

Below are the input parameters for the instances. All parameters are generated randomly with a uniform distribution. The data files used are available from the authors, upon request.
$c1_{ij} = t1_{ij} \times U[0.05, 0.1]$, unit transportation cost between supplier and DC
$c2_{jk} = t2_{jk} \times U[0.05, 0.1]$, unit transportation cost from DC to CDZ
$f_j = U[150,000, 160,000]$, fixed cost for installing a DC at location j
$g_j = U[0.01, 0.1]$, annual variable cost coefficient for installing DC at location j
$h1_j = U[0.1, 1]$, unit inventory holding cost at DC
$h2_k = U[0.1, 1]$, unit inventory holding cost at CDZ
$R_k = 0$, maximum guaranteed service time to customers at CDZ k
$SI_i = U[1, 5]$(integers), guaranteed service time of plant i
$t1_{ij} = U[1, 7]$(integers), order processing time of DC j if it is served by supplier i
$t2_{jk} = U[1, 3]$(integers), order processing time of CDZ k if it is served by DC j
$\mu_k = U[75, 150]$, daily mean demand at CDZ k
$\sigma_k^2 = U[0, 50]$, daily variance of demand at CDZ k
$\chi = 365$, days per year
$\theta1_{ij} = U[0.1, 1]$, annual unit cost of pipeline inventory from plant i to DC j
$\theta2_{jk} = U[0.1, 1]$, annual unit cost of pipeline inventory from DC j to CDZ k
$\lambda1_j = 1.96$, safety stock factor of DC j
$\lambda2_k = 1.96$, safety stock factor of CDZ k

Binary Variables

$X_{ij} = 1$ if DC j is served by supplier i
$Y_j = 1$ if a DC is installed at location j

$Z_{jk} = 1$ if the CDZ is served by DC j

Continuous Variables

L_k = net lead time of CDZ k
N_j = net lead time of DC j
S_j = guaranteed service time of DC j to the CDZs

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