

# Accelerating Benders Stochastic Decomposition for the Optimization under Uncertainty of the Petroleum Product Supply Chain

F. Oliveira<sup>a,\*</sup>, I.E. Grossmann<sup>b</sup>, S. Hamacher<sup>a</sup>

<sup>a</sup>*Industrial Engineering Department - Pontifícia Universidade Católica do Rio de Janeiro, Rua Marquês de São Vicente, 225, Gávea - Rio de Janeiro, RJ 22451-900, Brazil*

<sup>b</sup>*Chemical Engineering Department - Carnegie Mellon University, 5000 Forbes Ave, Pittsburgh, PA 15213, USA*

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## Abstract

This paper addresses the solution of a two-stage stochastic programming model for a supply chain investment planning problem applied to the petroleum products supply chain. In this context, we present the development of acceleration techniques for the stochastic Benders decomposition that aim to strengthen the cuts generated, as well as to improve the quality of the solutions obtained during the execution of the algorithm. Computational experiments are presented for assessing the efficiency of the proposed framework. We compare the performance of the proposed algorithm with two other acceleration techniques. Results suggest that the proposed approach is able to efficiently solve the problem under consideration, achieving better performance better in terms of computational times when compared to other two techniques.

*Keywords:* Stochastic programming, Supply chain investment pPlanning, Stochastic Benders decomposition, acceleration techniques

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\*Corresponding Author

*Email address:* hamacher@puc-rio.br (S. Hamacher)

## 1. Introduction

Currently, the use of large-scale, complex mixed integer linear programming (MILP) in the context of investment planning for supply chain problems is becoming more widespread. Even though commercial software can provide a good platform for solving many such problems, there are cases where conventional solvers cannot yield good solutions in a practically reasonable amount of time, especially when uncertainties are taken into consideration. Specifically, there is a trade-off between how precise is the representation of the uncertainties and the computational tractability of stochastic programming problems. On the one hand, one might need a large number of scenarios in order to accurately represent the uncertainties. On the other hand, it might turn out to be computationally infeasible to deal with a large number of scenarios by solving deterministic equivalents of stochastic programming problems.

One possible alternative to deal with this drawback is the use of a cutting-plane scheme that allows one to decompose the problem scenario-wise. Cutting-plane schemes has been successfully used in solving both large-scale deterministic and stochastic problems since the pioneering paper of Geoffrion and Graves [8], e.g., the uncapacitated network design problem with undirected arcs [10], stochastic transportation-location problems [7], locomotive and car assignment problems [4], stochastic supply chain design problems [24, 28], stochastic scheduling and planning of process systems [21, 31] and the stochastic unit commitment problem [18], to name a few.

The combination of Benders algorithm principles and stochastic problems is commonly referred to as the stochastic Benders decomposition, or also commonly referred to as the L-Shaped Method [29]. In this context, the decomposition is carried out by decomposing the complete deterministic equivalent problem [3] into a Master Problem (MP), which comprises the complicating variables and related constraints, and a Slave Problem, (SP) which is represented by the recourse decisions.

Nevertheless, under certain conditions, the traditional Benders decomposition (and consequently its stochastic version) might fail to achieve the aforementioned efficiency, a fact that has been broadly mentioned in the literature (see, for example Rei et al. [19], Saharidis et al. [23]). To circumvent this drawback various strategies have been proposed for accelerating Benders decomposition. McDaniel and Devine [12] proposed

the generation of cuts using the solution of a relaxed MP, and relaxing its integrality constraints. Furthermore, the authors present heuristic rules for determining when the integrality constraint is needed in order to ensure convergence of the algorithm. Although the results appear promising, the classical Benders decomposition can be more efficient in some cases. Cote and Laughton [5] showed another approach for accelerating Benders algorithms. In their approach, the MP is not solved to optimality but only the first integer solution obtained is used to generate the optimality or feasibility cut from the SP. The main drawback of this strategy is that by generating only cuts associated with suboptimal solutions, the algorithm may fail to generate cuts that are necessary to ensure convergence.

Within the context of generating more effective cuts, most researchers have sought to generate a set of cuts additional “strong” cuts at each iteration, or by modifying the way that Benders cuts are generated. Magnanti and Wong [11] proposed a seminal procedure for generating Pareto-optimal cuts to strengthen the standard Benders optimality cuts, though with the often challenging requirement of identifying, and even updating, a core point, which is required to lie inside the relative interior of the convex hull of the problem subregion defined in terms of MP variables. Papadakos [16] highlights that the Magnanti-Wong’s cut generation problem dependency on the solution of SP can sometimes decrease the algorithms performance. To circumvent this difficulty, the author showed that one can obtain an independent formulation of the Magnanti-Wong cut generation problem by dropping the constraint that implies in the former dependency on the solution of the subproblem. The author also provided guidelines for efficiently generating additional core points through convex combinations of previously known cores points and feasible solutions of the MP. More recently, Sherali and Lunday [25] presented a different strategy for generating non-dominated cuts through the use of preemptively small perturbation on the right-hand-side of the SP to generate maximal non-dominated Benders cuts. The authors also showed a strategy based on complimentary slackness that simplifies the cut generation when compared with the traditional strategy used by Magnanti and Wong [11]. Saharidis and Ierapetritou [22] proposed the generation of an additional valid Benders cut based on a maximum feasible subsystem whenever a Benders feasibility cut is generated. These cuts were shown to significantly accelerate

the convergence for problems where the number of feasibility cuts generated is greater than the number of optimality cuts. Fischetti et al. [6] presented an alternative scheme that combines the generation of Benders cuts when both optimality and feasibility cuts are required. They formulate a subproblem using an auxiliary variable that scales the subproblem as well as the resulting cut, where the generated cut acts as optimality and feasibility cuts according to whether the variable takes on positive or zero values. [19] investigate how local branching techniques can be used to accelerate Benders algorithm by applying local branching throughout the solution process. The authors also showed how Benders feasibility cuts can be strengthened or replaced with local branching. Saharidis et al. [23] examined two applications of a scheduling problem, for which they demonstrated the effectiveness of generating covering cut bundles to enhance Benders cuts.

The objectives of this paper are twofold. First, we present a framework for solving a two-stage stochastic programming model for a supply chain investment planning problem applied to petroleum products based on stochastic Benders decomposition. Second, we present the development of acceleration techniques tailored for the proposed approach. The proposed techniques addresses two different aspects in terms of algorithmic acceleration, since they aim at generating stronger cuts for the Benders decomposition in the context of stochastic programming, and they apply techniques for improving the quality of solutions obtained during the algorithm execution.

This paper is organized as follows: Section 2 presents the problem considered in this study and the mathematical formulation used. Section 3 describes the Stochastic Benders decomposition framework applied to the proposed model. Section 4 shows the application of the acceleration techniques considered to the problem under study. Section 5 describes how the computational experiments were performed and reports the numerical results obtained. Finally, we draw some conclusions in Section 6.

## 2. Problem Formulation

We consider an integrated distribution network design, facility location and discrete capacity expansion problem under a multi-product and multi-period setting, arising in the context of transportation planning of petroleum products distribution companies

operating over large geographical regions.

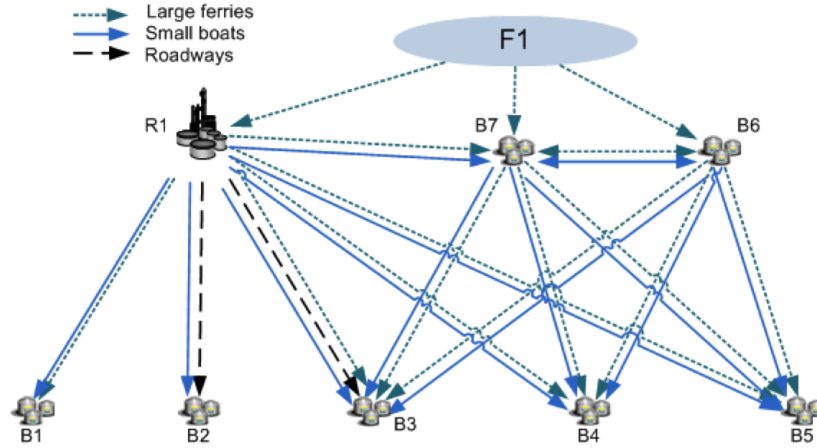


Figure 1: Example of a network

As shown in Figure 2, a supply chain is given composed of suppliers (represented by source F1 and refinery R1) and demand points (represented by points B1 to B7), which are connected through capacitated transportation arcs representing different transportation modes (in this case, three modes are considered namely large ferries, small boats and roadways). The demand points, which we will denote hereafter as distribution bases, are allowed to store product if necessary, given that the problem is considered under a multi-period setting. The storage and throughput capacities of the bases are limited, although they can be improved through the implementation of expansion projects. The same idea holds for arcs, which can also be expanded in the same fashion. In addition to that, we also consider the possibility of building new arcs and bases.

The objective of the plan is to determine which of the possible investments should be implemented and when, together with the transportation and inventory decisions that will cope with the projected (although uncertain) growth of product demands while minimizing both investment and logistics present costs.

To address the problem in question, a two-stage stochastic programming model is proposed [3]. The first-stage comprises the decisions of which projects to implement and when; the second-stage decisions are those relating to the flows of products, inventory levels, supply provided to each demand site, and supply levels at sources. The purpose of

the model is to determine the optimal investment selection product distribution to meet the demand of bases, minimizing both investment and logistics costs. The transportation decisions are defined in conjunction with investment decisions, which are chosen from a predefined portfolio of possibilities and allocated over the planning horizon. The uncertainties in the model are related to demand levels, which are modeled as random variables.

The notation used to state the following model is given in Appendix A. The mathematical model for the optimization of the aforementioned problem can be stated as follows:

$$\min_{w, y \in \{0,1\}} \sum_{j,p,t} W_{jpt} w_{jpt} + \sum_{i,j,t} Y_{ijt} y_{ijt} + \mathcal{Q}(w, y) \quad (2.1)$$

s.t.:

$$\sum_t w_{jpt} \leq 1 \quad \forall j \in \mathcal{B}, p \in \mathcal{P} \quad (2.2)$$

$$\sum_t y_{ijt} \leq 1 \quad \forall i, j \in \mathcal{N} \quad (2.3)$$

where  $w_{jpt}$  represent the capacity expansion investment decisions at location  $j$  for product  $p$  and in period  $t$  and  $y_{ijt}$  are connecting locations  $i$  and  $j$  investment decisions in period  $t$ ,  $\mathcal{Q}(w, y) = \mathbb{E}_\Omega[\mathcal{Q}(w, y, \xi)]$  represents the expectation evaluated over all  $\xi \in \Omega$  possible realizations for the uncertain parameters of the second-stage problem, given an investment decision  $(w, y)$ . Constraints 2.2 and 2.3 define that each investment can happens only once along the time horizon considered.

The second-stage problem  $\mathcal{Q}(w, y)$  can be stated as shown in equations 2.4 to 2.10. The objective function 2.4 represents freight costs between nodes, inventory costs, and cost of shortfall. Equation 2.5 comprises the material balance in distribution bases. Constraint 2.6 limits the supply availability at sources. Constraint 2.7 defines the arc capacities and the possibility of its expansion through the investment decisions. In a similar way, constraint 2.8 defines the storage capacities together with its expansion possibility. Constraint 2.9 defines minimum inventory levels, according to safety requirements. Constraint 2.10 sets the throughput limit for bases, defined by the product of the available

storage capacity with the throughput capacity multiplier.

$$\min_{x,v,u \in \mathbb{R}^+} \sum_{\xi} P^{\xi} \left( \sum_{i,j,p,t} C_{ijt} x_{ijpt}^{\xi} + \sum_{j,p,t} H_{jpt} v_{jpt}^{\xi} + \sum_{j,p,t} S_{jpt} u_{jpt}^{\xi} \right) \quad (2.4)$$

s.t.:

$$\sum_i x_{ijpt}^{\xi} + v_{jpt-1}^{\xi} + u_{jpt}^{\xi} = \sum_i x_{ijpt}^{\xi} + v_{jpt}^{\xi} + D_{jpt}^{\xi} \quad \forall j \in \mathcal{B}, p \in \mathcal{P}, t \in \mathcal{T}, \xi \in \Omega \quad (2.5)$$

$$\sum_j x_{ijpt}^{\xi} \leq O_{ipt} \quad \forall i \in \mathcal{S}, p \in \mathcal{P}, t \in \mathcal{T}, \xi \in \Omega \quad (2.6)$$

$$\sum_p x_{ijpt}^{\xi} \leq A_{ij}^0 + A_{ij} \sum_{t' \leq t} y_{ijt'} \quad \forall i, j \in \mathcal{N}, t \in \mathcal{T}, \xi \in \Omega \quad (2.7)$$

$$v_{jpt}^{\xi} \leq M_{jp}^0 + M_{jp} \sum_{t' \leq t} w_{jpt'} \quad \forall j \in \mathcal{B}, p \in \mathcal{P}, t \in \mathcal{T}, \xi \in \Omega \quad (2.8)$$

$$v_{jpt}^{\xi} \geq L_{jp} \left( M_{jp}^0 + M_{jp} \sum_{t' \leq t} w_{jpt'} \right) \quad \forall j \in \mathcal{B}, p \in \mathcal{P}, t \in \mathcal{T}, \xi \in \Omega \quad (2.9)$$

$$\sum_i x_{ijpt}^{\xi} \leq K_{jp} \left( M_{jp}^0 + M_{jp} \sum_{t' \leq t} w_{jpt'} \right) \quad \forall j \in \mathcal{B}, p \in \mathcal{P}, t \in \mathcal{T}, \xi \in \Omega \quad (2.10)$$

### 3. Stochastic Benders Decomposition

The model proposed in the previous section can be defined as an optimization model with first-stage integer variables and second-stage continuous variables. Such characteristics allow us to consider a scenario-wise decomposition framework based on Benders decomposition [1] applied to stochastic optimization, given the particular diagonal structure of these problem, where the first-stage variables arise as complicating in a sense that they are the only elements providing connection between each scenario subproblem [29]. Moreover, the model has complete recourse [3], that is, for any feasible first-stage solution, the second stage problem is always feasible. Note that this fact is convenient since it precludes the generation of feasibility cuts in order to ensure feasibility in the context of the stochastic Benders decomposition.

We start by noting that the MP can be equivalently reformulated as follows:

$$\min_{w,y \in \{0,1\}} \sum_{j,p,t} W_{jpt} w_{jpt} + \sum_{i,j,t} Y_{ijt} y_{ijt} + M \quad (3.1)$$

s.t.:

$$\sum_t w_{jpt} \leq 1 \quad \forall j \in \mathcal{B}, p \in \mathcal{P} \quad (3.2)$$

$$\sum_t y_{ijt} \leq 1 \quad \forall i, j \in \mathcal{N} \quad (3.3)$$

$$M \geq \mathcal{Q}(w, y) \quad (3.4)$$

This formulation allows one to distinguish an important issue. Inequality 3.4 cannot be used computationally as a constraint since it is not defined explicitly, but only implicitly by a number of optimization problems. The main idea of the proposed decomposition method is to relax this constraint and replace it by a number of cuts, which may be gradually added following an iterative solving process. These cuts, defined as supporting hyperplanes of the second-stage objective function, might eventually provide a good estimation for the value of  $\mathcal{Q}(w, y)$  in a finite number of iterations. In order to define the form of these cuts, let us first state the dual formulation of the second-stage problem, which represents the SP in our context, given a first-stage solution  $(\bar{w}, \bar{y})$ :

$$\begin{aligned} \mathcal{Q}(\bar{w}, \bar{y}) &= \mathbb{E}_\Omega[\mathcal{Q}(\bar{w}, \bar{y}, \xi)] = \\ & \max_{\alpha, \beta, \gamma, \delta, \zeta, \eta} \sum_\xi P^\xi \left( \sum_{j,p,t} D_{jpt}^\xi \alpha_{jpt}^\xi + \sum_{j,p,t} O_{jpt} \beta_{jpt}^\xi + \right. \\ & \sum_{i,j,t} \left( A_{ij}^0 + A_{ij} \sum_{t' \leq t} \bar{y}_{ijt'} \right) \gamma_{ijt}^\xi + \\ & \sum_{j,p,t} \left( M_{jp}^0 + M_{jp} \sum_{t' \leq t} \bar{w}_{jpt} \right) \\ & \left. \left( \delta_{jpt}^\xi + L_{jp} \eta_{jpt}^\xi + K_{jp} \zeta_{jpt}^\xi \right) \right) \end{aligned} \quad (3.5)$$

s.t.:

$$\alpha_{jpt}^\xi - \alpha_{ipt}^\xi + \beta_{ipt}^\xi + \gamma_{jpt}^\xi + \zeta_{jpt}^\xi \leq C_{ijt} \quad \forall i, j \in \mathcal{N}, p \in \mathcal{P}, t \in \mathcal{T}, \xi \in \Omega \quad (3.6)$$

$$\alpha_{jpt}^\xi - \alpha_{jpt-1}^\xi + \delta_{jpt}^\xi + \eta_{jpt}^\xi \leq H_{jpt} \quad \forall j \in \mathcal{B}, p \in \mathcal{P}, t \in \mathcal{T}, \xi \in \Omega \quad (3.7)$$

$$\alpha_{jpt}^\xi \leq S_{jpt} \quad \forall j \in \mathcal{S}, p \in \mathcal{P}, t \in \mathcal{T}, \xi \in \Omega \quad (3.8)$$

Let  $\Pi$  denote the set of all extreme points of the polyhedron defined by the feasible space of the Dual Slave Problem (DSP) given by 3.5 to 3.8 and  $\alpha, \beta, \gamma, \delta, \zeta$ , and  $\eta$  denote the dual variables associated with constraints 2.5 to 2.10, and  $M$  the objective function value.



Also, letting  $M^*$  be the optimal value, we must have  $M^* \geq M^{(k)}, \forall k \in \Pi$ . Therefore, our DSP can be restated as  $\mathcal{Q}(\bar{w}, \bar{y}) = \min_{M \geq 0} \{M : M \geq M^{(k)}, \forall k \in \Pi\}$ , where

$$\begin{aligned}
M^{(k)} = & \sum_{\xi \in \Omega} P^\xi \left( \sum_{j,p,t} D_{jpt}^\xi \alpha_{jpt}^{\xi(k)} + \sum_{j,p,t} O_{jpt} \beta_{jpt}^{\xi(k)} + \right. \\
& \sum_{i,j,t} \left( A_{ij}^0 + A_{ij} \sum_{t' \leq t} y_{ijt'} \right) \gamma_{ijt}^{\xi(k)} + \\
& \left. \sum_{j,p,t} \left( M_{jp}^0 + M_{jp} \sum_{t' \leq t} w_{jpt} \right) \left( \delta_{jpt}^{\xi(k)} + L_{jp} \eta_{jpt}^{\xi(k)} + K_{jp} \zeta_{jpt}^{\xi(k)} \right) \right) \quad (3.9)
\end{aligned}$$

Using the above representation for the DSP that is based on the extreme points  $k \in \Pi$  of its polyhedron, we can now replace equation 3.4 with the new reformulation 3.9 for  $\mathbb{E}_\Omega[Q(w, y, \xi)]$  in the MP, providing the following:

$$\min_{w, y \in \{0,1\}} \sum_{j,p,t} W_{jpt} w_{jpt} + \sum_{i,j,t} Y_{ijt} y_{ijt} + M \quad (3.10)$$

s.t.:

$$\sum_t w_{jpt} \leq 1 \quad \forall j \in \mathcal{B}, p \in \mathcal{P} \quad (3.11)$$

$$\sum_t y_{ijt} \leq 1 \quad \forall i, j \in \mathcal{N} \quad (3.12)$$

$$\begin{aligned}
M \geq & \sum_{\xi \in \Omega} P^\xi \left( \sum_{j,p,t} D_{jpt}^\xi \alpha_{jpt}^{\xi(k)} + \sum_{j,p,t} O_{jpt} \beta_{jpt}^{\xi(k)} + \right. \\
& \sum_{i,j,t} \left( A_{ij}^0 + A_{ij} \sum_{t' \leq t} y_{ijt'} \right) \gamma_{ijt}^{\xi(k)} + \\
& \left. \sum_{j,p,t} \left( M_{jp}^0 + M_{jp} \sum_{t' \leq t} w_{jpt} \right) \left( \delta_{jpt}^{\xi(k)} + L_{jp} \eta_{jpt}^{\xi(k)} + K_{jp} \zeta_{jpt}^{\xi(k)} \right) \right) \quad \forall k \in \Pi \quad (3.13)
\end{aligned}$$

This reformulation has the drawback of comprising a very large number of constraints of type 3.13. Moreover, at the optimal solution, not all of the constraints in 3.13 will be binding. Therefore, in the iterative Benders decomposition algorithm, one works with a relaxed version of MP by considering only a subset of 3.13 at each iteration. We denote this subset by  $\Pi'$  which includes the constraints 3.13 generated via solving the DSP in the previous iterations. This relaxed formulation of the MP (RMP) considering only the subset  $\Pi'$  of cuts in 3.13 provides a lower bound to the optimal solution of the MP.

At a given iteration of the stochastic Benders decomposition, a RMP<sup>(k)</sup> is solved first to obtain the values of  $(\bar{w}^{(k)}, \bar{y}^{(k)})$ . Then, these values are used to solve DSP<sup>(k)</sup> to obtain the values of dual variables  $\alpha^{(k)}, \beta^{(k)}, \gamma^{(k)}, \delta^{(k)}, \zeta^{(k)}, \eta^{(k)}$  (i.e., an extreme point  $k \in \Pi$  of the dual polyhedron) and a new cut of the form 3.13 to include into  $\Pi'$ . Note that when the DSP<sup>(k)</sup> is solved for given  $(\bar{w}^{(k)}, \bar{y}^{(k)})$ , an upper bound for MP can be easily calculated by adding DSP<sup>(k)</sup>'s objective value and the total fixed cost component for the RMP<sup>(k)</sup> (i.e., the objective value of RMP<sup>(k)</sup> excluding  $M^{(k)}$ ).

#### 4. Accelerating Benders Decomposition

In this section we present the techniques developed for speeding-up the decomposition framework presented in the previous section.

##### 4.1. Multi cut framework

Recall that in the stochastic Benders decomposition presented in the previous section, a single optimality cut is added at each iteration. This cut aims to approximate the value of the second-stage function at the current solution. However, instead of using only a single cut at each iteration, one can add multiple cuts to approximate the individual second-stage cost function corresponding to each one of the  $|\Omega|$  scenarios. In this case, the RMP can be reformulated as follows:

$$\min_{w, y \in \{0,1\}} \sum_{j,p,t} W_{jpt} w_{jpt} + \sum_{i,j,t} Y_{ijt} y_{ijt} + M^\xi \quad (4.1)$$

s.t.:

$$\sum_t w_{jpt} \leq 1 \quad \forall j \in \mathcal{B}, p \in \mathcal{P} \quad (4.2)$$

$$\sum_t y_{ijt} \leq 1 \quad \forall i, j \in \mathcal{N} \quad (4.3)$$

$$\begin{aligned} M^\xi \geq & \sum_{j,p,t} D_{jpt}^\xi \alpha_{jpt}^{\xi(k)} + \sum_{j,p,t} O_{jpt} \beta_{jpt}^{\xi(k)} + \\ & \sum_{i,j,t} \left( A_{ij}^0 + A_{ij} \sum_{t' \leq t} y_{ijt'} \right) \gamma_{ijt}^{\xi(k)} + \\ & \sum_{j,p,t} \left( M_{jp}^0 + M_{jp} \sum_{t' \leq t} w_{jpt'} \right) \left( \delta_{jpt}^{\xi(k)} + L_{jp} \eta_{jpt}^{\xi(k)} + K_{jp} \zeta_{jpt}^{\xi(k)} \right) \quad \forall \xi \in \Omega, k \in \Pi' \end{aligned} \quad (4.4)$$

Birge and Louveaux [2] showed that the use of such a framework may greatly speed-up convergence. The main idea behind this multi cut framework is to generate an outer linearization for all functions  $Q(w, y, \xi)$ , replacing the outer linearization of  $\mathcal{Q}(w, y)$ . The multi cut approach relies on the idea that using independent outer approximations of all functions  $Q(w, y, \xi)$  provide more information to the MP than the single cut on  $\mathcal{Q}(w, y)$ , and therefore, fewer iterations are needed to reach the optimal solution.

#### 4.2. Generating stronger cuts

Magnanti and Wong [11] proposed a seminal methodology to accelerate convergence of Benders decomposition by strengthening the generated cuts. They observed that in certain cases where the SP presents degeneracy, one might generate different cuts for the same optimal solution  $(\bar{w}^{(k)}, \bar{y}^{(k)})$ , each one of different strength in terms of efficiently approximating the second-stage cost function. To circumvent this difficulty, the authors proposed a methodology for identifying the strongest possible cut, which they referred to as the Pareto-optimal cut.

To illustrate the technique, assume that we have our problem written in the following form:

$$\min\{c^T z + q^T r \mid Az + Br \geq h, z \in Z\} \quad (4.5)$$

where  $Z \subseteq \mathbb{R}^n$  represents linear constraints involving the first-stage  $n$  variables and their integrality requirements,  $r \in \mathbb{R}^p$ ,  $A$  and  $B$  are  $m \times n$  and  $m \times p$  matrices,  $c \in \mathbb{R}^n$ ,  $q \in \mathbb{R}^p$ , and  $h \in \mathbb{R}^m$ .

Similarly to what we did in Section 3, let  $\Pi = \{\pi \in \mathbb{R}^m \mid B^T \pi \leq q\}$  be the set of feasible solutions for DSP. In addition, let  $M \geq \pi^T (h - Az)$  represent a Benders cut, and  $\Pi_{alt}$  be the set of alternative optimal solutions for the DSP given a MP solution  $\bar{z}$ . Then, we say that the Benders cut generated by  $\pi^*$  dominates all others (i.e., is Pareto-optimal) if:

$$\pi^{*T} (h - A\bar{z}) \leq \pi^T (h - A\bar{z}) \quad \forall \pi \in \Pi_{alt} \quad (4.6)$$

Magnanti and Wong [11] showed how one can generate Pareto-optimal cuts based on the notion of core points. A core point is defined as a point  $\hat{z}$  in the relative interior of  $Conv(Z)$ , where  $Conv(\cdot)$  denotes the convex hull [30]. They proved that if a cut is

selected such that it attains the maximum value at a core point amongst the set of all alternative cuts, then this cut is not dominated by other cuts at any feasible solution - a Pareto-optimal cut.

In order to generate these cuts, they suggest selecting some core point  $\hat{z}$ , and after solving SP for  $\bar{z}$  (which we denote hereafter as  $SP(\bar{z})$ ), they generate the Benders cut by subsequently solving a secondary subproblem, which can be stated as follows:

$$\max\{\pi^T(h - A\hat{z}) \mid \pi \in \Pi_{alt}\} \quad (4.7)$$

where  $\Pi_{alt} = \{\pi \in \Pi \mid \pi^T(h - A\bar{z}) = f(SP(\bar{z}))\}$  is the set of alternative optimal solutions of 4.7 and  $f(\cdot)$  the objective function value.

Nevertheless, there are certain implementation issues related to Magnanti and Wong's cut generation procedure. First, the dependency of the subproblem 4.7 to the DSP might jeopardize the algorithm efficiency, especially in the cases where the DSP might turn out to be difficult to solve. Moreover, since Magnanti and Wong's procedure requires a new core point at each major iteration (recall that the core points rely inside the convex hull of the MP, whose feasible region is changing at each major iteration), it might be the case that it is not easy to obtain new core points at each iteration. To address this drawback, researchers often approximate core points [24, 17], arbitrarily define it by fixing components of the core point vector [13] or use alternative points derived from a given problem structure [16]. In addition to that, it is important to note that this strategy does not always yield a net computational advantage since the trade-off between the reduction in the number of iterations required compared to the increase in the number of linear programs solved to generate each cut might not pay-off [14].

In this paper we propose an alternative way of generating nondominated cuts based on the definition of maximal cuts. Following the ideas of Sherali and Lunday [25] for generating maximal nondominated Benders cuts, we show an alternative for strengthening the Benders Cuts while circumventing the aforementioned drawbacks.

First, we start by highlighting the standard definition of maximal, typically used in cutting plane theory from integer programming literature [30]. Let us rewrite the Benders cut generated from a selected  $\pi \in \Pi_{alt}$  as:

$$M \geq \pi^T h + \sum_{j=1}^n (-\pi^T A_j) \bar{z}_j \quad (4.8)$$

Then, we say that a Benders cut is maximal for a given  $\pi$  if, for every  $\pi' \in \Pi_{alt}$ , we have that  $\pi^T h \geq \pi'^T h$  and  $-\pi^T A_j \geq -\pi'^T A_j$ . It is not difficult to see that a Pareto-optimal or nondominated cut generated in the way proposed by Magnanti and Wong [11] can be also considered maximal, provided that the core point  $\hat{z}$  is positive.

Sherali and Lunday [25] show that the aforementioned concept of maximal cuts can be used to derive an alternative way of generating cuts that would accelerate Benders decomposition. To achieve such a goal, we must first view the process of generating maximal cuts as one of determining a Pareto-optimal solution to the multiple objective problem defined as:

$$\max\{\pi^T h, -\pi^T A_1, \dots, -\pi^T A_n \mid \pi \in \Pi_{alt}\} \quad (4.9)$$

We can obtain a solution for this problem by selecting a positive weight vector (PWV) and then maximizing the positive weighted sum of the multiple functions in 4.9 [27]. By doing this, we end up by having to solve the following problem:

$$\max\{\pi^T h + \sum_{j=1}^n -\pi^T A_j \hat{z} \mid \pi \in \Pi_{alt}\} \quad (4.10)$$

which is exactly the problem defined in 4.7. Therefore, if we define  $\hat{z}$  as a positive core point solution, then the resulting cut would be both maximal as well as nondominated.

Seeking to obtain an efficient framework to derive these cuts, we can combine in the same problem both the step where we solve DSP( $\bar{z}$ ) to obtain  $\Pi_{alt}$  and the subsequent step of solving 4.7. Toward this end, we must first note that we are essentially considering a priority multiple objective program, where we want to first maximize DSP( $\bar{z}$ ) (i.e.,  $\pi^T(h - A\bar{z})$  subject to  $\pi \in \Pi$ ) and next, considering all alternative solutions to this problem, choose the one which maximizes  $\pi^T(h - A\hat{z})$ . Again, one might note that the approach of Magnanti and Wong [11] to generate nondominated cuts using the core point  $\hat{z}$  can be interpreted in the same way.

Sherali and Soyster [26] showed that such a multiple objective program can be equivalently solved by the following combined weighted sum problem:

$$\max\{\pi^T(h - A\bar{z}) + \mu\pi^T(h - A\hat{z}) \mid \pi \in \Pi\} \quad (4.11)$$

where  $\mu$  is a suitably small weight. Although Sherali and Soyster [26] showed that it is always possible to derive  $\mu$  such that it would render 4.11 equivalent to the multi-

objective problem mentioned before, the derivation of such a weight is not typically a practically convenient task except in some particular cases.

In order to circumvent this drawback, we propose an alternative way of dealing with the weight  $\mu$  in order to obtain what we call as *dynamically updated near-maximal Benders cuts*. The main reasoning behind the following ideas are rather experimental than theoretical. What we observe from our numerical experiments is that the solutions obtained in the early iterations yield poor descriptions of the second-stage cost curve, which is exactly what we are trying to approximate through the use of Benders cuts in the stochastic programming context. Moreover, we observe that by applying the aforementioned ideas of generating maximal cuts, we can consider 4.11 as an auxiliary problem to simulate the existence of more dense first-stage solutions in the early iterations in order to speed-up the convergence. As for the algorithmic procedure, we can then iteratively adjust the weight  $\mu$  in order to favor solutions that are more focused on improving the original  $DSP(\bar{z})$  objective value  $\pi^T(h - A\bar{z})$  rather than 4.11.

One important characteristic of such a framework for updating the weight  $\mu$  is that it does not prevent convergence if a proper sequence  $\{\mu^{(k)}\}_{k=1,\dots,\infty}$  is selected. In order to keep the original convergence properties of the traditional Benders decomposition, it follows that one might select a sequence of  $\mu_b$ ,  $b = 1, \dots, \infty$  such that the following properties hold:

1.  $\sum_{k=1}^{\infty} \mu^{(k)} \rightarrow \infty$
2.  $\mu^{(k)} \rightarrow 0$  as  $k \rightarrow \infty$

By selecting such a divergent series, it is not difficult to see that convergence is guaranteed since:

$$\lim_{\mu \rightarrow 0} \pi^T(h - A\bar{z}) + \mu \pi^T(h - A\hat{z}) = \pi^T(h - A\bar{z}) \quad (4.12)$$

allowing us to rely on the results from Benders [1] (or from Van Slyke and Wets [29] for the stochastic version, or even from Birge and Louveaux [2] for the multi cut framework), which guarantees convergence for the algorithm.

#### 4.3. Additional acceleration ideas

Combined with the strengthening of the cuts generated at each major iteration of the proposed algorithm, we also use additional acceleration ideas in order to improve

computational efficiency.

#### 4.3.1. Upper bound improving

In our implementation of the stochastic Benders decomposition, we observed that there is a strong relationship between the quality of the incumbent solutions  $(\bar{w}, \bar{y})$  obtained during the execution and the convergence rate of the algorithm. This issue is related with the fact that, especially in the early iterations, the incumbent solutions obtained may be quite far from the optimal solution, leading the algorithm to explore inferior parts of the feasible region. However, if a good solution  $(\bar{w}, \bar{y})$  is made available through the use of some heuristic, we can use it in place of the incumbent solution and proceed from there. Our algorithm makes use of a particular heuristic in order to try to generate these good solutions during the algorithm execution.

The heuristic relies on facts observed during our computational experiments. We observed that, after the optimality gap becomes reasonably small, the bounds exhibit a tailing off behavior as the iterations progress. This effect is mainly due to the fact that, in these iterations, all the incumbent solution tend to present identical or very similar selection of projects (in terms of location and product for  $w_{jpt}$  and origin and destination for  $y_{ijt}$ ), only changing timing decisions. Because the timing decisions have relatively small influence on the objective value, the upper bound changes very little. In order to avoid this behavior, this heuristic after a certain number of iterations with no improvement on the upper bound. The heuristic consists of three main steps:

1. Fix the current project selection to those in the current incumbent solution.
2. Randomly sample a subset of the scenario set  $\Omega$  and solve the equivalent deterministic to determine whether this selection investment should be selected indeed, and if so, when. Notice that by doing this, we are both reducing the size of the second-stage problem (since it is a subset of  $\Omega$ ), as well as the size of the first-stage problem (since we are only considering investments decided in terms of location and product for  $w_{jpt}$  and origin and destination for  $y_{ijt}$ , hence considering fewer integer variables).
3. Evaluate the obtained solution to check if it provides an improved upper bound. If so, use this solution to update the incumbent solution and the correspondent upper bound as the incumbent upper bound.

### 4.3.2. Trust-region

As pointed out by Ruszczyński [20], the initial iterations of decomposition methods based on cutting planes tend to present an unstable behavior. This effect is mainly due to the fact that the solutions tend to oscillate between different sections of the feasible region, what may lead to slow convergence.

In the continuous case, this effect can be effectively controlled by the use of two different approaches. The first consists of adding a regularizing term in the MP objective function that penalizes the  $l_2$ -distance between the current solution and the previous one [20]. The second focuses on constraining the  $l_\infty$  distance of the MP variables from the previous solution within a trust region [9]. These extensions prevent the MP solution from moving far from the previous iterate. One point that must be highlighted is that both the penalty magnitude and the size of the trust region must be controlled during the execution of the algorithm based on its progress. Using a proper control is imperative when using these techniques in order to avoid losing convergence properties.

In our problem, the first-stage variables are binary vectors. In this case, using a  $l_2$  regularizing term would render a mixed-integer quadratic MP, which would become much more complex in terms of solution methodology. Moreover, a  $l_\infty$  trust-region would be useless in our case. Since feasible MP solutions are extreme points of the unit hypercube, a trust region of size greater or equal than one would include all its vertices (i.e., all possible binary feasible solutions), while a trust-region with size less than one would include only the previous solution.

Santoso et al. [24] show how one can deal with this drawback by using the Hamming distances between the binary solution vector as a measuring unit for the trust region. Let  $(\bar{w}^{(k)}, \bar{y}^{(k)})$  be the MP in iteration  $k$  and let  $\mathcal{W} = \{(j, p, t) \mid \bar{w}_{jpt}^{(k)} = 1\}$  and  $\mathcal{Y} = \{(i, j, t) \mid \bar{y}_{ijt}^{(k)} = 1\}$ . Then, we impose the following constraint in the MP to be solved in iteration  $k + 1$ :

$$\begin{aligned} & \sum_{(j,p,t) \in \mathcal{W}} (1 - w_{jpt}) + \sum_{(i,j,t) \in \mathcal{Y}} (1 - y_{ijt}) + \\ & \sum_{(j,p,t) \notin \mathcal{W}} w_{jpt} + \sum_{(i,j,t) \notin \mathcal{Y}} y_{ijt} \leq \Gamma^{(k+1)} \end{aligned} \tag{4.13}$$

where  $\Gamma^{(k+1)} < (|\mathcal{N} \times \mathcal{P} \times \mathcal{T}| + |\mathcal{N} \times \mathcal{N} \times \mathcal{T}|)$  represents the trust-region size in itera-



tion  $k + 1$ . Unfortunately, convergence cannot be guaranteed if a non-redundant trust region is used throughout the algorithm execution. Hence, since the algorithm tends to have the oscillating effect that we are willing to avoid mostly in the beginning of the execution, we dynamically adjust the size as the algorithm converges. When the algorithm reaches a sufficiently small optimality gap, we drop 4.13 from the MP in order to ensure convergence.

## 5. Computational Experiments

This section describes the computational experiments performed to evaluate the proposed algorithm under different considerations. All experiments described in this section were executed in an Intel Xeon 2.4GHz CPU with 4GB RAM and implemented in AIMMS 3.12. The mixed-integer and linear programming models within the decomposition framework were solved with CPLEX 12.3.

In order to assess the efficiency of the proposed framework, we consider two different instances. The first consists of a small example formulated to illustrate the efficiency of the proposed approach. The second example consists of a realistic case study of a large-scale investment planning problem. We test both instances comparing the results with three different techniques. The first technique used (for now on referred as *Algorithm 1*) generates nondominated Benders cuts according to Magnanti and Wong [11], with the approximation and core point updating technique as proposed by Papadakos [16], i.e., we initialize a core point approximation  $\hat{z}$  with a feasible solution to the MP and then update the approximation at each successive iteration by setting  $\hat{z} = \lambda \hat{z} + (1 - \lambda)\bar{z}$ . We adopt  $\lambda = 0.5$ , as prescribed by Papadakos [16]. The author states that, based on empirical observation, such a value for  $\lambda$  usually yields better results in terms of algorithmic convergence. The second technique (*Algorithm 2*) used consists of generating maximal nondominated Benders cut as proposed by Sherali and Lunday [25]. The authors show that one can use the following expression to calculate  $\mu$  that yields a near-optimal maximal Benders cut:

$$\mu = \frac{\epsilon_0}{M\theta} \tag{5.1}$$

where  $\epsilon$  is a prespecified tolerance on the absolute optimality gap,  $M$  is the penalty for

recourse unfeasibility (that is equivalent to the cost of unmet demand in our case), and  $\theta = \epsilon_0 + \max\{0, \max\{\hat{h}_i\}\} - \min\{0, \min\{\hat{h}_i\}\}$ , with  $\hat{h} = h - A\hat{z}$ . We used a fixed value for the weight  $\mu$ , as shown by the authors in their calculations, considering  $\epsilon_0$  as being 1% of the true optimal value for Example 1. In Example 2, we empirically derived a fixed value for  $\mu$ , since it is not practically convenient to base the selection of  $\epsilon_0$  on the optimal solution. Note that by doing that, we are in effect implicitly dictating a particular choice of  $\epsilon_0$ . Finally, *Algorithm 3* use the *dynamically updated near maximal Benders cuts* proposed in section 4.2 as the cut generation strategy. In both cases we update the multipliers according to the following divergent series:

$$\mu^{t+1} = \frac{k}{l|B|} \mu^t \tag{5.2}$$

where  $k$  and  $l$  are pre specified parameters, and  $|B'|$  represents the number of iterations up to now. In both experiments we use  $k = 2$ ,  $l = 1$ , and  $\mu_0$  is given by 5.1. We use  $\epsilon_0$  as 5% of the true optimal value for Example 1 and an empirically fixed value for Example 2.

### 5.1. Example 1

The first example is a simplified version of the investment planning model introduced in Section 2. It involves capacity expansions and network design decisions for a simplified network composed by three supply sites, five primary bases (intermediate markets which works also as distribution points), and four secondary bases (which are equivalent to final markets) as seen in Figure 2. We consider five time periods and sixteen scenarios for the demand at both primary and secondary bases. For generating the scenarios we consider that the demand in each region can either grow 5% per year, or decrease 1% per year. We consider that the example comprises five different regions: B1 and B2 as Region 1, B3 as Region 2, B4 and B5 as Region 3, C1 and C2 as Region 4, and C3 and C4 as Region 5. The data considered for this experiment is given in Appendix B.

The equivalent deterministic model has 80 binary variables, 2803 continuous variables, and 3859 constraints. The optimal solution of this problem is \$1,520.47 million over a five-year period. The optimal first-stage solution is illustrated in Figure 2. The arcs between two different regions (represented by gray background rectangles) are the arcs that can be selected to compose the network design, while the second row of nodes

represents the primary bases, all of them possible candidates for capacity expansion. The enlarged spheres represents the bases that had their capacity expanded and the solid arrows between two different regions represent arcs that were selected to compose the network design. The model decides for two capacity expansions from the investment portfolio composed by five options (B4 in period 1 and B2 in period 2) and for the selection of eight arcs, among ten options provided (B1-C1, B2-C2, B3-C2, B3-C3, B4-C3, and B5-C4 in period 1, B2-C1 in period 3, and B4-C4 in period 4).

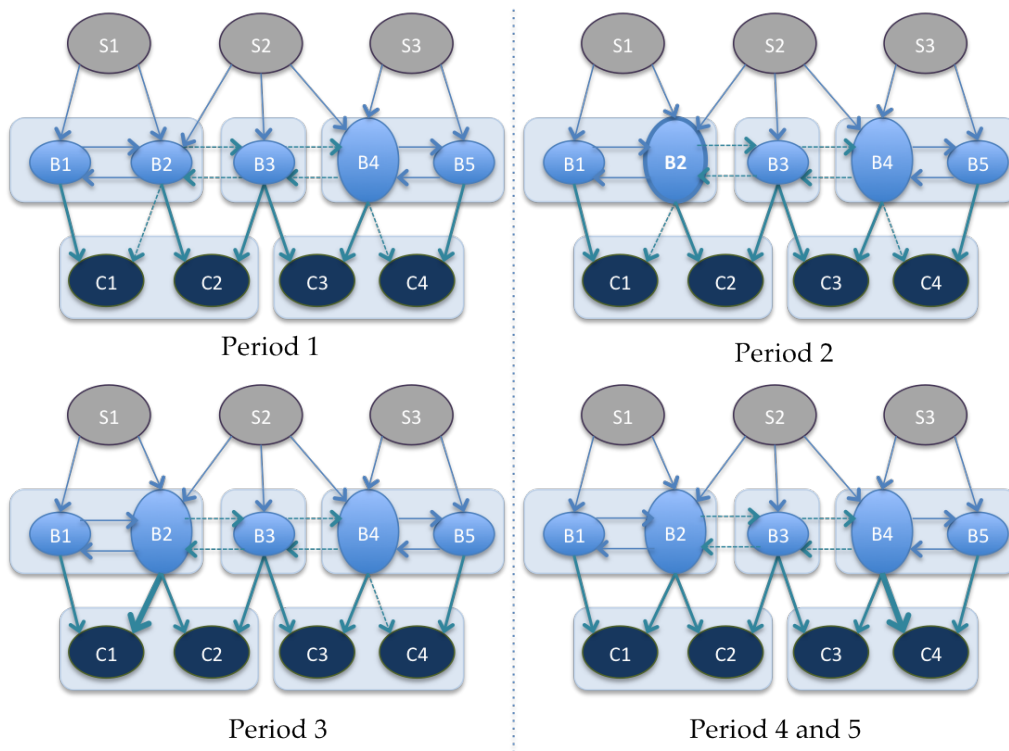


Figure 2: Graphical representation of first-stage solution

of this first example is to provide numerical evidence that the proposed cut strengthening technique provides stronger cuts for the stochastic Benders decomposition, hence improving convergence. In all algorithms we use the multi cut formulation, as presented in Section 4.1. In addition, we do not use any of the additional acceleration techniques presented in Sections 4.3.1 and 4.3.2, since they did not provide any improvement in this case. All algorithms were solved considering a convergence criterion of 0.01% gap

between upper and lower bounds.

Table 1 summarizes the performance of each cut generation technique, compared with the use of traditional Benders cuts. The computational results allow us to draw two important observations. First, one can notice that all acceleration techniques improved the algorithm efficiency in terms of number of iterations to reach the convergence criterion, and consequently, in terms of computational time, when compared with the traditional Benders cut generation procedure. Second, one can observe that, even though the three algorithms provide similar results, the proposed approach (*Algorithm 3*) is able to reach the optimal solution slightly faster.

|                 | Traditional Cuts | <i>Algorithm 1</i> | <i>Algorithm 2</i> | <i>Algorithm 3</i> |
|-----------------|------------------|--------------------|--------------------|--------------------|
| # of iterations | 35               | 21                 | 26                 | 19                 |
| CPU Time[s]     | 10.93            | 5.90               | 7.22               | 4.86               |

Table 1: Example 1 result summary

## 5.2. Example 2

The second example represents a realistic supply chain investment planning problem under demand uncertainty. The transportation can in principle be performed in different modes, but in this case study it is primarily done using waterways, which are strongly affected by seasonality issues regarding the navigability of rivers. Four different products were considered - diesel, gasoline, aviation fuel and fuel oil - to be distributed over 19 locations (13 bases and 1 refinery and 2 external supply locations). The portfolio of projects considered for the study consists of 28 local projects and three arc projects. The planning horizon considered was 6 years, divided into a total of 24 quarterly periods. All experiments in this example were solved up to 2% optimality gap. For further details on this instance, please refer to Oliveira and Hamacher [15].

To take into account the uncertainty in demand levels for petroleum products, scenarios were generated by the following first order autoregressive model:

$$D_{jpt} = D_{jpt-1} (1 + \omega_p + \sigma\varepsilon) \quad (5.3)$$

where  $\omega_p$  represents the expected average growth rate for the consumption of product  $p$  over the planning horizon,  $\sigma$  represents the estimated maximum deviation for product

consumption in the region and  $\varepsilon$  follows a standard normal distribution. The average growth rate, as well as the maximum deviation were forecasted based on the analysis of the annual consumption historical series over the last 40 years. Each scenario represents a possible demand curve for the whole time horizon considered, for each product and place.

In order to assess the efficiency of the proposed approach, we performed two different experiments. The first experiments seeks to illustrate the effects of using the acceleration techniques proposed in sections 4.3.1 (Upper bounding improving) and 4.3.2(Trust regions) in a random sample composed by 200 scenarios. Table 2 summarizes the performance of these techniques compared them individually and in combination with the case where none of such acceleration techniques are used. Note that in Table 2 the bounds are not reported at their last iteration since we are using arbitrary increments in the number of iterations.

| # Iter.     | No Acc.   |        | UB Imp.   |      | TR        |      | UB Imp. + TR |      |
|-------------|-----------|--------|-----------|------|-----------|------|--------------|------|
|             | UB        | %gap   | UB        | %gap | UB        | %gap | UB           | %gap |
| 1           | 1732310.5 | 99.2   | 1732310.5 | 99.2 | 1732310.5 | 99.2 | 1732310.5    | 99.2 |
| 5           | 481802.2  | 46.3   | 481802.2  | 46.4 | 481802.2  | 46.4 | 481802.2     | 46.3 |
| 10          | 418213.5  | 7.3    | 416152.5  | 10.1 | 416152.5  | 3.4  | 418213.5     | 7.3  |
| 15          | 412056.4  | 3.7    | 408188.4  | 2.5  | 411950.4  | 2.5  | 408188.4     | 2.4  |
| 20          | 412056.4  | 2.7    | -         | -    | 411950.4  | -    | -            | -    |
| 25          | 412056.4  | 2.3    | -         | -    | -         | -    | -            | -    |
| 30          | 412056.4  | 2.1    | -         | -    | -         | -    | -            | -    |
| CPU Time[s] |           | 1151.2 | 617.7     |      | 619.0     |      | 446.3        |      |

Table 2: Example 2 result summary - experiment 1

Table 2 allows us to observe the effect of the acceleration techniques proposed in Sections 4.3.1 and 4.3.2 separately (indicated as UB Imp. and TR, respectively). From the results we can conclude that both techniques improve convergence of the proposed algorithm, reducing the total CPU time required to reach the convergence criterion by approximately 46% in both cases. Moreover, when we combine both techniques, the reduction in the solution times is even larger, yielding improvement on the solution time

of over 61%.

For the second experiment in this example, we developed a set consisting of 100 independent scenario samples of sizes varying from 20 to 200 scenarios, as can be seen in Table 3. Notice that in this case, we are solving the problem for 1000 different demand instances, since the samples are independent. Our objective by doing that is to assess the efficiency of the proposed approach independent of particularities of a given scenario sample.

| Scenarios | <i>Algorithm 1</i> |          | <i>Algorithm 2</i> |          | <i>Algorithm 3</i> |          | CPLEX |          |
|-----------|--------------------|----------|--------------------|----------|--------------------|----------|-------|----------|
|           | Avg.               | St. Dev. | Avg.               | St. Dev. | Avg.               | St. Dev. | Avg.  | St. Dev. |
| 20        | 56.5               | 19.2     | 55.5               | 15.0     | 22.3               | 2.2      | 9.6   | 0.6      |
| 40        | 104.7              | 37.7     | 102.1              | 31.3     | 43.2               | 2.7      | 33.7  | 3.3      |
| 60        | 162.4              | 66.5     | 140.7              | 52.0     | 64.0               | 4.5      | 51.5  | 5.0      |
| 80        | 219.5              | 78.6     | 221.5              | 111.8    | 83.7               | 5.8      | 104.0 | 20.9     |
| 100       | 287.2              | 124.3    | 272.5              | 139.2    | 109.1              | 12.7     | 185.6 | 34.5     |
| 120       | 369.6              | 150.5    | 313.9              | 152.4    | 128.3              | 9.2      | 287.8 | 46.3     |
| 140       | 387.6              | 163.3    | 381.8              | 170.9    | 151.2              | 10.8     | 473.0 | 122.3    |
| 160       | 483.8              | 208.8    | 372.3              | 135.8    | 176.7              | 11.6     | 601.1 | 140.8    |
| 180       | 612.7              | 270.3    | 477.1              | 189.4    | 203.8              | 13.9     | 734.9 | 165.6    |
| 200       | 631.4              | 252.7    | 521.8              | 196.1    | 230.1              | 12.2     | 927.6 | 161.0    |

Table 3: Example 2 result summary of CPU-times - experiment 2

Table 3 presents the statistical data retrieved from the experiments carried out in Example 2, showing the average time solution (Avg. column) and the standard deviation (Std. Dev. column) in CPU seconds. As can be seen in Table 3, the CPLEX times are smaller in the experiments when a small number of scenarios is considered. However, it is important to keep in mind that these examples with smaller number of scenarios are only for completeness of the experimental set since they are not capable of giving a good description of the uncertainty considered (for further information on sample size specifications, please refer to Oliveira and Hamacher [15]). As the number of scenarios increases, the decomposition frameworks outperform the use of CPLEX. We also highlight the performance of our algorithm (*Algorithm 3*) in the case study under consideration.

As we can observe from the experimental results, the proposed algorithm performed better than the other cutting generation strategy in all experiments. In addition, for cases where more than 80 scenarios were considered, *Algorithm 3* reached the best average solution times among all solution procedures, including CPLEX.

Another remarkable feature that can be observed in Table 3 is related to the standard deviation of the solution times. The results suggest that *Algorithm 3* attains the smaller deviation in terms of CPU seconds regarding the time the algorithm takes to reach a 2% gap suboptimal solution. The observation of this fact leads us to the conclusion that the performance of our *Algorithm 3* is less affected by particular characteristics of the scenario sample itself since it seems to be more robust in terms of solution time variation.

## 6. Conclusions

In this paper we have presented the development of acceleration techniques for the stochastic Benders decomposition to solve an investment planning problem applied to the petroleum products supply chain. We have proposed a new methodology for generating dynamically updated near-maximal Benders cuts, and compared it with acceleration techniques proposed by Papadakos [16] and Sherali and Lunday [25] for the stochastic Benders algorithm. Moreover, we have proposed the application of two additional acceleration techniques to further improve the convergence of the algorithm, especially in cases where convergence is difficult due to the computational complexity of the problem at hand.

We conducted two numerical examples to assess the efficiency of the proposed framework. In Example 1 we showed how the acceleration technique can be applied to a small problem. The results provide some numerical evidences regarding the benefits one could obtain by using the proposed acceleration technique. In Example 2 we tackled a large-scale real world problem. Since we are dealing with uncertainty through the use of a sampling framework, we choose to generate a large number of instances (100 samples of 10 different sizes) by repeatedly sampling a first order autoregressive model. The problem considered represents a realistic supply chain investment planning problem considering four different products to be distributed over 19 locations (13 bases, 1 refinery and 2 external supply locations). As our computational results suggest, our algorithm performed

faster for this particular problem considered under a sampling framework. The experimental results show that, for a larger number of scenarios, the proposed algorithm can perform 4.5 times faster on average than solving the full-space equivalent deterministic problem. Moreover, our algorithm also presented better results when compared to other acceleration approaches considered.

As for future research, we are seeking to further evaluate the technique under a more broader context, testing these ideas for more general types of problems commonly used in the literature. We also believe that the development of more exhaustive sensitivity analysis of the algorithmic performance to variations in user-specified parameters deserve further examination in a future study.

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## Appendix A. Nomenclature

### Indexes and sets

|                        |              |
|------------------------|--------------|
| $i, j \in \mathcal{N}$ | Locations    |
| $p \in \mathcal{P}$    | Products     |
| $t \in \mathcal{T}$    | Time periods |
| $\xi \in \Omega$       | Scenarios    |

### Subsets

|                                     |                    |
|-------------------------------------|--------------------|
| $\mathcal{B} \subseteq \mathcal{N}$ | Distribution bases |
| $\mathcal{S} \subseteq \mathcal{N}$ | Suppliers          |

### Parameters

|               |                                |
|---------------|--------------------------------|
| $A_{ij}^0$    | Current arc capacity           |
| $A_{ij}$      | Additional arc capacity        |
| $C_{ijt}$     | Transportation cost            |
| $D_{jpt}^\xi$ | Demand                         |
| $H_{jpt}$     | Inventory cost                 |
| $K_{jp}$      | Throughput capacity multiplier |
| $M_{jp}^0$    | Current inventory capacity     |
| $M_{jp}$      | Additional inventory capacity  |
| $O_{jpt}$     | Supply                         |
| $S_{jpt}$     | Shortfall cost                 |
| $W_{jpt}$     | Inventory investment cost      |
| $Y_{ijt}$     | Arc investment cost            |

### Variables

|                |                               |
|----------------|-------------------------------|
| $x_{ijpt}^\xi$ | Product flow                  |
| $v_{jpt}^\xi$  | Inventory level               |
| $u_{jpt}^\xi$  | Unmet demand                  |
| $w_{jpt}$      | Inventory investment decision |
| $y_{ijt}$      | Arc investment decision       |

## Appendix B. Data of the Experiment 1

| Arc   | Freight Cost [\$/m <sup>3</sup> ] | Investment Cost[1000\$] |
|-------|-----------------------------------|-------------------------|
| B1 C1 | 0.1                               | 11                      |
| B2 B1 | 0                                 | 20                      |
| B2 B3 | 1.2                               | 15                      |
| B2 C1 | 0                                 | 17                      |
| B2 C2 | 0                                 | 13                      |
| B3 B2 | 1.6                               | 13                      |
| B3 B4 | 1.8                               | 19                      |
| B3 C2 | 0                                 | 16                      |
| B3 C3 | 0                                 | 16                      |
| B4 B3 | 1.1                               | 15                      |
| B4 B5 | 0.2                               | -                       |
| B4 C3 | 0                                 | 13                      |
| B4 C4 | 0                                 | 12                      |
| B5 B4 | 0.3                               | -                       |
| B5 C4 | 0                                 | 10                      |

Table B.5: Example 1 data: Arc freight and investment cost

| Base | Initial Capacity<br>[1000m <sup>3</sup> ] | Expansion Capacity<br>[1000m <sup>3</sup> ] | Investment Cost<br>[1000\$] | Inventory Cost<br>[\$/m <sup>3</sup> ] |
|------|---|---|-----------------------------|--|
| B1   | 40  | 8   | 752                         | 0.5                                    |
| B2   | 30  | 6   | 528                         | 0.7                                    |
| B3   | 40  | 8   | 712                         | 1                                      |
| B4   | 30  | 12  | 996                         | 0.2                                    |
| B5   | 50  | 10  | 800                         | 1                                      |

Table B.6: Example 1 data: locations initial capacity, expansion capacity, investment cost, and inventory cost

|                                       |     |
|---------------------------------------|-----|
| Supply Limit [m <sup>3</sup> ]        | 100 |
| Arc Capacity [1000m <sup>3</sup> ]    | 50  |
| Throughput Capacity Adjustment Factor | 1.5 |
| Return Rate                           | 8%  |
| Safety Level                          | 10% |

Table B.7: Example 1 data: general parameters

| Base | Initial Demand [1000m <sup>3</sup> ] |
|------|--------------------------------------|
| B1   | 30                                   |
| B2   | 25                                   |
| B3   | 30                                   |
| B4   | 40                                   |
| B5   | 30                                   |
| C1   | 10                                   |
| C2   | 15                                   |
| C3   | 20                                   |
| C4   | 15                                   |

Table B.8: Example 1 data: base initial demands