Symmetry breaking generalized disjunctive formulation for the strip packing problem.

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Abstract

a

1 Introduction

The two-dimensional strip packing is a special case of "Cutting and Packing" problems that arises in many applications. The problem seeks to pack a given set of rectangles into a strip of given width in order to minimize its length. Some applications in which the strip packing problem arises are: Cutting pieces from wooden boards, cutting pieces from glass or steel sheets, optimal layout of industrial facilities, etc. Lodi et al.[1] present a comprehensive survey of application and methods. Following the typology proposed by Wascher et al.[2] for cutting and packing problems, the two-dimensional strip packing is classified as two-dimensional open dimension problem (2-ODP).

Although there exist several heuristic and meta heuristic algorithms for solving the problem[3], exact algorithms have been proposed for the solution of the two-dimensional strip packing. Martello et al.[4] present a branch and bound algorithm. The lower bound of the nodes in this method is obtained considering the geometry of the problem. Alvarez-Valdes et al.[5] improve the branch and bound method of Martello et al. by obtaining stronger lower bounds and deriving new dominance conditions.

In addition to specialized algorithms, the two-dimensional strip packing problem has been formulated as a mixed-integer linear program (MILP). Westerlund et al.[6] present an MILP model for the N-dimensional allocation, which includes the strip packing problem. Castro and Oliveira[7] present two different MILP formulations for the problem, based on scheduling models. One formulation follows a discrete-space approach, and it is based on the Resource-Task Network process representation. The other formulation uses a hybrid continuous/discrete representation, in which the width of the strip is discrete and the height is treated as continuous.

Generalized disjunctive programming (GDP) is an alternative framework form modelling discrete-continuous optimization problems[8]. GDP models can be reformulated as MILP (MINLP) models, or solved with specialized GDP algorithms[9]. The GDP-to-MILP reformulations are traditionally the big-M (BM)[10] and Hull Reformulation (HR)[11]. An improved Big-M reformulation (MBM) is presented by Trespalacios and Grossmann[12]. Sawaya and Grossmann[13] present a GDP model for the two-dimensional strip packing.

In this work we present an alternative GDP model for the strip packing problem. This model, which is a modification of the model by Sawaya and Grossmann, uses additional constraints within the disjunctions in order to break symmetry in the solutions. The model is further improved for the case in which the heights and lengths of the rectangles are integer (which is true in general with a simple transformation, as long as the lengths and heights of the rectangles are rational numbers). The model is reformulated with the three GDP-to-MILP schemes (BM, MBM, and HR) and solved as an MILP. The new formulations with the alternative GDP-to-MILP reformulations are tested with 100 random instances. The results show that the new symmetry-breaking formulation is solved faster than the formulation by Sawaya and Grossmann. They also show that the improved Big-M reformulation (MBM) solves faster than the traditional GDP-to-MILP reformulations (BM and HR).

2 GDP formulation of the two-dimensional strip packing problem

The two-dimensional strip packing problem consists on placing a given set of N rectangles in a strip. The height and length of each rectangle is known ($H_i, L_i; i \in N$), and the strip has width W. The rectangles cannot be rotated. The objective is to minimize the total length of the strip. Figure 1 illustrates the strip packing problem.

The GDP formulation of this problem, presented by Sawaya and Grossmann[13] is as follows:

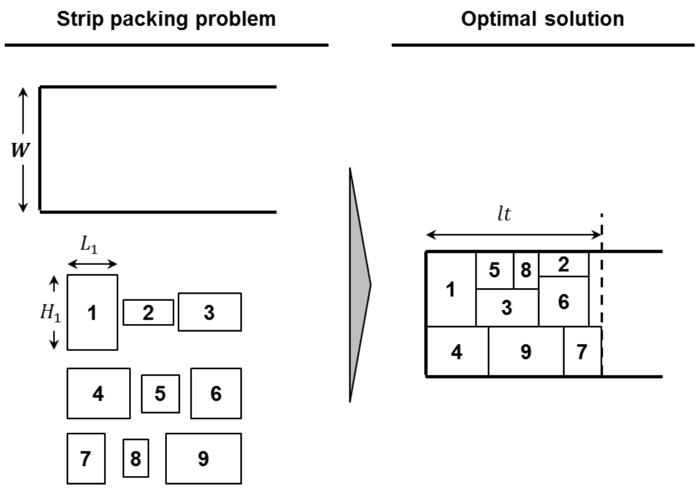

Figure 1: Illustration of the two-dimensional strip packing problem.

$$
\begin{aligned}
& min\ lt \\
s.t. \quad & lt \geq x_i + L_i && i \in N \\
& \begin{bmatrix} Z^1_{ij} \\ x_i + L_i \leq x_j \end{bmatrix} \vee \begin{bmatrix} Z^1_{ji} \\ x_j + L_j \leq x_i \end{bmatrix} \\
& \vee \begin{bmatrix} Z^2_{ij} \\ y_i - H_i \geq y_j \end{bmatrix} \vee \begin{bmatrix} Z^2_{ji} \\ y_j - H_j \geq y_i \end{bmatrix} && i, j \in N, i < j \qquad \text{(SG)} \\
& Z^1_{ij} \veebar Z^1_{ji} \veebar Z^2_{ij} \veebar Z^2_{ji} && i, j \in N, i < j \\
& 0 \leq x_i \leq UB - L_i && i \in N \\
& H_i \leq y_i \leq W && i \in N \\
& Z^1_{ij}, Z^2_{ij} \in \{True, False\} && i, j \in N, i \neq j
\end{aligned}
$$

In (SG), the continuous variables (x_i, y_i) represent the coordinates of the upper-left corner of rectangle i. The objective is to minimize the distance lt. The global constraints ($lt \geq x_i + L_i$) enforce that the total distance is greater than the x coordinate of the right edge of each rectangle. The logic variables indicate the relative position between two rectangles. $Z^1_{ij} = True$ if rectangle i is to the left of rectangle j. $Z^2_{ij} = True$ if rectangle i is on top of rectangle j. The disjunction establishes the four possible relative positions between each pair of rectangles (i.e. rectangle i is to the left, right, top or bottom of rectangle j). Exactly one alternative must be selected ($Z^1_{ij} \veebar Z^1_{ji} \veebar Z^2_{ij} \veebar Z^2_{ji}$). For the selected alternative the corresponding constraints are enforced (e.g. if $Z^1_{ij} = True$ then $x_i + L_i \leq x_j$ is enforced). For the terms no selected (e.g. $Z^1_{ji} = False$) the corresponding constraints are ignored. The continuous variables have upper and lower bounds, where UB is an upper bound for the length of the strip (a simple upper bound can be obtained with $UB = \sum_i L_i$).

Figure 2 represents the disjunction in the formulation. The figure shows the possible positions for a

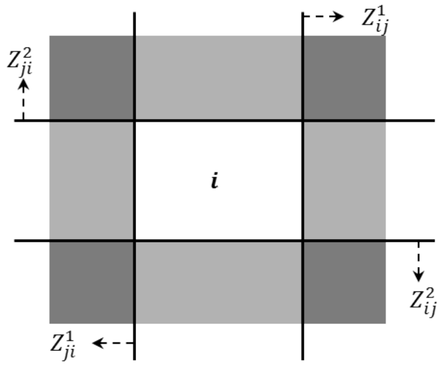

Figure 2: Illustration of the relative position of j with respect to rectangle i.

rectangle j with respect to a rectangle i. Note that there are two alternative decisions that are feasible for the darker regions of the figure. For example, if j is both above and to the right of i it is possible to set $Z_{ij}^1 = True; Z_{ji}^1 = Z_{ij}^2 = Z_{ji}^2 = False$ or $Z_{ji}^2 = True; Z_{ji}^1 = Z_{ji}^1 = Z_{ij}^2 = False$. If $Z_{ij}^1 = True$ the constraints corresponding to Z_{ji}^2 are ignored (which means that they may or may not be satisfied).

It is possible to reformulate (SG) as an MILP and solve it with an MILP solver. There are three possible reformulations: (BM)[10], (MBM)[12] and (HR)[11]. For details on formulating GDP problems and reformulating as MILP or MINLP, we refer the reader to the review work by Grossmann and Trespalacios[14].

For this problem, the (BM) and (MBM) reformulations are the same. The resulting MILP is as follows:

$$
\begin{aligned}
& min\ lt & & \\
s.t.\quad & lt \geq x_i + L_i & & i \in N \\
& x_i + L_i \leq x_j + UB(1 - z_{ij}^1) & & i,j \in N, i < j \\
& x_j + L_j \leq x_i + UB(1 - z_{ji}^1) & & i,j \in N, i < j \\
& y_i - H_i \geq y_j - W(1 - z_{ij}^2) & & i,j \in N, i < j \\
& y_j - H_j \geq y_i - W(1 - z_{ji}^2) & & i,j \in N, i < j \\
& z_{ij}^1 + z_{ji}^1 + z_{ij}^2 + z_{ji}^2 = 1 & & i,j \in N, i < j \\
& 0 \leq x_i \leq UB - L_i & & i \in N \\
& H_i \leq y_i \leq W & & i \in N \\
& z_{ij}^1, z_{ij}^2 \in \{0,1\} & & i,j \in N, i \neq j
\end{aligned}
\tag{SG-BM}
$$

In (SG-BM), the Boolean variables (Z_{ij}^1, Z_{ij}^2) are transformed to binary variables with a one-to-one correspondence (i.e. $z_{ij}^1 = 1$ is equivalent to $z_{ij}^1 = True$, while $z_{ij}^1 = 0$ is equivalent to $z_{ij}^1 = False$). Exactly one disjunctive term must be selected $(z_{ij}^1 + z_{ji}^1 + z_{ij}^2 + z_{ji}^2 = 1)$. When a disjunctive term is selected, the corresponding constraints are enforced (e.g. If $z_{ij}^1 = 1$, then $x_i + L_i \leq x_j$). When it is not selected, the corresponding constraints become redundant (e.g. If $z_{ji}^1 = 0$, then $x_j + L_j \leq x_i + UB$).

The (HR) reformulation of this problem is as follows:

$$
\begin{aligned}
& min\ lt & & \\
s.t.\quad & lt \geq x_i + L_i & i \in N & \\
& x_i = \nu_i^{ij} + \nu_i^{ji} & i, j \in N, i \neq j & \\
& y_i = \mu_i^{ij} + \mu_i^{ji} & i, j \in N, i \neq j & \\
& \nu_i^{ij} + L_i z_{ij}^1 \leq \nu_j^{ij} & i, j \in N, i < j & \\
& \nu_j^{ji} + L_j z_{ji}^1 \leq \nu_i^{ji} & i, j \in N, i < j & \\
& \mu_i^{ij} - H_i z_{ij}^2 \geq \mu_j^{ij} & i, j \in N, i < j & \\
& \mu_j^{ji} - H_j z_{ji}^2 \geq \mu_i^{ji} & i, j \in N, i < j & \qquad \text{(SG-HR)} \\
& z_{ij}^1 + z_{ji}^1 + z_{ij}^2 + z_{ji}^2 = 1 & i, j \in N, i < j & \\
& 0 \leq \nu_i^{ij} \leq (UB - L_i) z_{ij}^1 & i, j \in N, i \neq j & \\
& 0 \leq \nu_i^{ji} \leq (UB - L_i) z_{ji}^1 & i, j \in N, i \neq j & \\
& H_i z_{ij}^2 \leq \mu_i^{ij} \leq (W) z_{ij}^2 & i \in N, i \neq j & \\
& H_i z_{ji}^2 \leq \mu_i^{ji} \leq (W) z_{ji}^2 & i \in N, i \neq j & \\
& 0 \leq x_i \leq UB - L_i & i \in N & \\
& H_i \leq y_i \leq W & i \in N & \\
& z_{ij}^1, z_{ij}^2 \in \{0, 1\} & i, j \in N, i \neq j &
\end{aligned}
$$

In (SG-HR), the Boolean variables (Z_{ij}^1, Z_{ij}^2) are transformed to binary variables, and exactly one disjunctive term must be selected ($z_{ij}^1 + z_{ji}^1 + z_{ij}^2 + z_{ji}^2 = 1$). The variables are disaggregated so that one variable is included for every disjunctive term in which it appears. When a disjunctive term is selected, the disaggregated variable must lie within the variable bounds (e.g. If $z_{ij}^1 = 1$, then $0 \leq \nu_i^{ij} \leq (UB - L_i)$ and $0 \leq \nu_j^{ij} \leq (UB - L_j)$). When a term is not selected, its corresponding disaggregated variable becomes zero (e.g. If $z_{ji}^1 = 0$, then $0 \leq \nu_i^{ji} \leq 0$ and $0 \leq \nu_j^{ji} \leq 0$). The constraints of the selected disjunctions are enforced to the disaggregated variables (e.g. If $z_{ij}^1 = 1$, then $\nu_i^{ij} + L_i \leq \nu_j^{ij}$). The constraints corresponding to not selected terms are trivially satisfied ($0 \leq 0$).

It is easy to see that (HR) is a larger formulation than (BM). However, it provides a formulation that is as strong, or stronger, than the (BM)[15].

3 Symmetry-breaking GDP formulation

It is clear from Figure 2 that some regions can be represented with the selection of two different disjunctive terms. Because of that, problem (SG-BM) has several symmetric solutions.

An alternative formulation to break some of the symmetry in the problem is as follows:

$$
\begin{aligned}
& min\ lt && \\
s.t. \quad & lt \geq x_i + L_i && i \in N \\
& \begin{bmatrix} Z^1_{ij} \\ x_i + L_i \leq x_j \end{bmatrix} \vee \begin{bmatrix} Z^1_{ji} \\ x_j + L_j \leq x_i \end{bmatrix} && \\
& \vee \begin{bmatrix} Z^2_{ij} \\ y_i - H_i \geq y_j \\ x_i + L_i \geq x_j \\ x_j + L_j \geq x_i \end{bmatrix} \vee \begin{bmatrix} Z^2_{ji} \\ y_j - H_j \geq y_i \\ x_i + L_i \geq x_j \\ x_j + L_j \geq x_i \end{bmatrix} && i, j \in N, i < j \\
& Z^1_{ij} \veebar Z^1_{ji} \veebar Z^2_{ij} \veebar Z^2_{ji} && i, j \in N, i < j \\
& 0 \leq x_i \leq UB - L_i && i \in N \\
& H_i \leq y_i \leq W && i \in N \\
& Z^1_{ij}, Z^2_{ij} \in \{True, False\} && i, j \in N, i \neq j
\end{aligned}
\tag{S1}
$$

The modified disjunction in (S1) separates the feasible region that corresponds to $Z^2_{ij} = True$ (and $Z^2_{ji} = True$) from the feasible region that corresponds to $Z^1_{ij} = True$ and to $Z^1_{ji} = True$. The disjunction is illustrated in Figure 3. In the figure, the pattern to represent the feasible region of $Z^1_{ij} = True$ is different from the pattern used to represent $Z^2_{ij} = True$. The reason for this is to illustrate that for $Z^1_{ij} = True$ the complete rectangle needs to be positioned in the grey region. In contrast, it is possible to have $Z^2_{ij} = True$ a as long as a part of the rectangle is in the striped region. The feasible region that corresponds to $Z^2_{ij} = True$ represents not only i above j, but also i is not to the right or to the left of j. Note that the strategy could be applied to an alternative formulation in which $Z^2_{ij} = True$ represents only that i is above j, and $Z^1_{ij} = True$ represents i left j and also i is not above or below j. Both of these formulation serve the purpose of breaking symmetric solutions, and in this work we will focus problem (S1).

Problem (S1) breaks some of the symmetry. However, if a rectangle j is on one side of rectangle i, but the edge of rectangle j is aligned to i, there is still symmetry. Figure 4 illustrates such a case. In the Figure, j is above i and not to the right or left or i (it satisfies the constraints corresponding to $Y^2_{ji} = True$). The figure shows that j is also to the right of i (it satisfies the constraints associated to $Y^1_{ji} = True$).

If H_i and L_i are integer, it is possible to break this symmetry with the following formulation (note that if H_i and L_i are rational numbers they can be transformed into integer values):

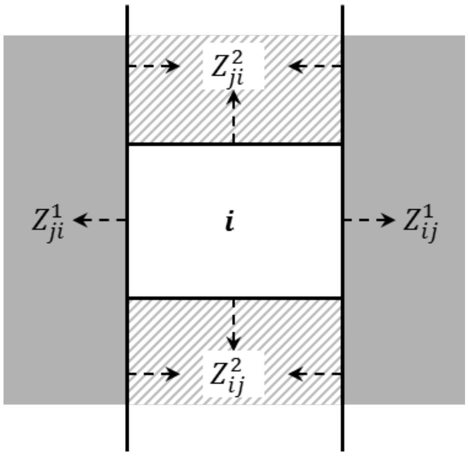

Figure 3: Illustration of the relative position of j with respect to rectangle i for the symmetry breaking formulation (S1).

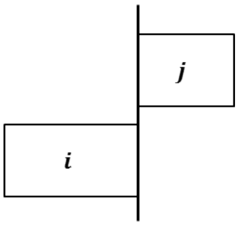

Figure 4: Illustration of symmetry in which either $Y_{ij}^1 = True$ or $Y_{ji}^2 = True$.

$$
\begin{array}{lll}
 & min\ lt & \\
s.t. & lt \geq x_i + L_i & i \in N \\
 & \begin{bmatrix} Z_{ij}^1 \\ x_i + L_i \leq x_j \end{bmatrix} \vee \begin{bmatrix} Z_{ji}^1 \\ x_j + L_j \leq x_i \end{bmatrix} & \\
 & \vee \begin{bmatrix} Z_{ij}^2 \\ y_i - H_i \geq y_j \\ x_i + L_i \geq x_j + 1 \\ x_j + L_j \geq x_i + 1 \end{bmatrix} \vee \begin{bmatrix} Z_{ji}^2 \\ y_j - H_j \geq y_i \\ x_i + L_i \geq x_j + 1 \\ x_j + L_j \geq x_i + 1 \end{bmatrix} & i, j \in N, i < j \\
 & Z_{ij}^1 \veebar Z_{ji}^1 \veebar Z_{ij}^2 \veebar Z_{ji}^2 & i, j \in N, i < j \\
 & 0 \leq x_i \leq UB - L_i & i \in N \\
 & H_i \leq y_i \leq W & i \in N \\
 & Z_{ij}^1, Z_{ij}^2 \in \{True, False\} & i, j \in N, i \neq j
\end{array} \tag{S2}
$$

Problem (S2) includes a "+1" in the right hand side of the constraints corresponding to Z_{ij}^2. The feasible region of the disjunction in this formulation is presented in Figure 5.

The (BM) and (HR) reformulations (S1) and (S2) are obtained in the same manner as (SG-BM) and (SG-

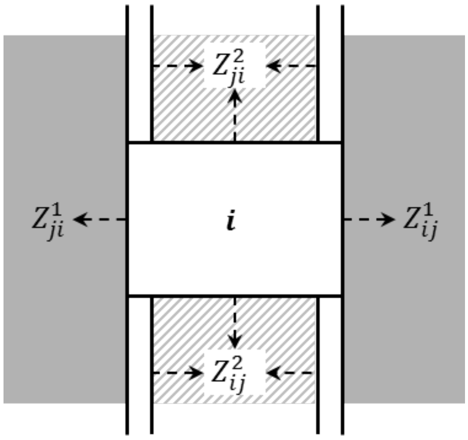

Figure 5: Illustration of the relative position of j with respect to rectangle i for the symmetry breaking formulation (S2).

HR) were obtained. We refer to the (BM) and (HR) formulation of (S1) as (S1-BM) and (S1-HR), and the (BM) and (HR) of (S2) as (S2-BM) and (S2-HR). However, in this case the (MBM) of (S1) and (S2) is not the same as the (BM). The (MBM) of (S1) is as follows:

$$
\begin{aligned}
& min\ lt && \\
s.t.\quad & lt \geq x_i + L_i && i \in N \\
& x_i + L_i \leq x_j + UBz^1_{ji} + (L_i + L_j)(z^2_{ij} + z^2_{ji}) && i,j \in N, i < j \\
& x_j + L_j \leq x_i + UBz^1_{ij} + (L_i + L_j)(z^2_{ij} + z^2_{ji}) && i,j \in N, i < j \\
& y_i - H_i \geq y_j - W(z^1_{ij} + z^1_{ji} + z^2_{ji}) && i,j \in N, i < j \\
& x_i + L_i \geq x_j - (UB - L_i - L_j)z^1_{ij} - (L_j - L_i)z^1_{ji} && i,j \in N, i < j \\
& x_j + L_j \geq x_i - (UB - L_i - L_j)z^1_{ji} - (L_i - L_j)z^1_{ij} && i,j \in N, i < j \\
& y_j - H_j \geq y_i - W(z^1_{ij} + z^1_{ji} + z^2_{ij}) && i,j \in N, i < j \\
& z^1_{ij} + z^1_{ji} + z^2_{ij} + z^2_{ji} = 1 && i,j \in N, i < j \\
& 0 \leq x_i \leq UB - L_i && i \in N \\
& H_i \leq y_i \leq W && i \in N \\
& z^1_{ij}, z^2_{ij} \in \{0,1\} && i,j \in N, i \neq j
\end{aligned}
\tag{S1-MBM}
$$

In (S1-MBM), multiple M-parameters are used for the formulation. When a term is selected, all other binary variables in that disjunction become zero (e.g. If $z^2_{ij} = 1$, then $z^1_{ij} = z^1_{ji} = z^2_{ji} = 0$). Because of this, when a disjunctive term is selected the corresponding constraints are enforced. For example, if $z^2_{ij} = 1$ then $x_i + L_i \geq x_j - (UB - L_i - L_j)0 - (L_j - L_i)0$, so $x_i + L_i \geq x_j$. Note that in (S1-MBM) the M-parameters are different and they depend on the selected disjunctive term. When a term is not selected its corresponding constraints become redundant, but the M-parameter of the constraint depends on the disjunctive term that is selected. For example, if $z_{ij}^2 = 1$ then the constraint that corresponds to Z_{ij}^1 becomes redundant as follows: $x_i + L_i \leq x_j + (0)UB + (L_i + L_j)(1)$, so $x_i \leq x_j + L_j$. However, if $z_{ji}^1 = 1$ then the constraint that corresponds to Z_{ij}^1 becomes redundant as follows: $x_i + L_i \leq x_j + UB$. Also note that the constraints representing "j is not to the right or to the left of i" are the same in $Z_{ij}^2 = True$ and $Z_{ji}^2 = True$. Because of this, the constraints in the (MBM) reformulation only appear once.

The (MBM) of (S2) can be obtained in a similar manner, and we refer to it as (S2-MBM).

4 Numerical results

The different formulations were tested 100 random instances of the strip packing problem. The range of values of the random parameters is as follows: N = 5-14; W = 10-20; L_i = 1-5; H_i = 2-5. All of the instances were solved using GAMS 24.3.3[16], using an Intel(R) Core(TM) i7 CPU 2.93 GHz and 4 GB of RAM.

Acknowledgments

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References

[1] Lodi, A.; Martello, S.; Monaci, M. Two-dimensional packing problems: A survey. *European Journal of Operational Research* **2002**, *141*, 241–252.

[2] Wäscher, G.; Haußner, H.; Schumann, H. An improved typology of cutting and packing problems. *European Journal of Operational Research* **2007**, *183*, 1109–1130.

[3] Lodi, A.; Martello, S.; Vigo, D. Heuristic and metaheuristic approaches for a class of two-dimensional bin packing problems. *INFORMS Journal on Computing* **1999**, *11*, 345–357.

[4] Martello, S.; Monaci, M.; Vigo, D. An exact approach to the strip-packing problem. *INFORMS Journal on Computing* **2003**, *15*, 310–319.

[5] Alvarez-Valdes, R; Parreno, F; Tamarit, J. A branch and bound algorithm for the strip packing problem. *OR spectrum* **2009**, *31*, 431–459.

[6] Westerlund, J.; Papageorgiou, L. G.; Westerlund, T. A MILP model for N-dimensional allocation. *Computers & Chemical Engineering* **2007**, *31*, 1702–1714.

[7] Castro, P. M.; Oliveira, J. F. Scheduling inspired models for two-dimensional packing problems. *European Journal of Operational Research* **2011**, *215*, 45–56.

[8] Raman, R.; Grossmann, I. E. Modelling and computational techniques for logic based integer programming. *Computers & Chemical Engineering* **1994**, *18*, 563–578.

[9] Trespalacios, F.; Grossmann, I. E. Review of Mixed-Integer Nonlinear and Generalized Disjunctive Programming Methods. *Chemie Ingenieur Technik* **2014**, *86*, 991–1012.

[10] Wolsey, L. A.; Nemhauser, G. L. *Integer and combinatorial optimization*; John Wiley & Sons, 2014.

[11] Lee, S.; Grossmann, I. E. New algorithms for nonlinear generalized disjunctive programming. *Computers & Chemical Engineering* **2000**, *24*, 2125–2141.

[12] Trespalacios, F.; Grossmann, I. E. Improved Big-M reformulation for generalized disjunctive programs. *Computers & Chemical Engineering* **2015**, *76*, 98–103.

[13] Sawaya, N. W.; Grossmann, I. E. A cutting plane method for solving linear generalized disjunctive programming problems. *Computers & chemical engineering* **2005**, *29*, 1891–1913.

[14] Grossmann, I. E.; Trespalacios, F. Systematic modeling of discrete-continuous optimization models through generalized disjunctive programming. *AIChE Journal* **2013**, *59*, 3276–3295.

[15] Balas, E. Disjunctive programming. *Annals of Discrete Mathematics* **1979**, *5*, 3–51.

[16] Brooke, A.; Kendrick, D.; Meeraus, A.; Raman GAMS, a Users Guide. *The Scientific Press* **1998**.