On the computational studies of deterministic global optimization of head dependent short-term hydro scheduling

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***Abstract*—This paper addresses the global optimization of the short term scheduling for hydroelectric power generation. A Mixed Integer Nonlinear Programming (MINLP) model is proposed for a cascade of hydro plants, each one with multiple turbines, characterized by a detailed representation of the net head of water and a nonlinear hydropower generation function. In addition, a simplified model is also proposed where only the linear coefficients of the forebay and tailrace polynomial functions are retained. A deterministic global optimization approach, denominated sHBB, is developed and its performance in four case studies is compared with a commercial solver for global optimization. The results show that the proposed approach is more efficient than the commercial solver in terms of finding a better solution with a smaller optimality gap, using less CPU time. The proposed method can also find alternative and potentially more profitable power production schedules. Significant insights were also obtained regarding the effectiveness of the employed relaxation strategies.**

***Index Terms*—Short term hydro scheduling, MINLP, global optimization.**

NOMENCLATURE

A. Indices and sets

i,k,I	Hydro plants
IC	Pairs of upstream and downstream plants
qq, G	Grid points for the relaxations of bilinear terms
j,J	Turbines
M	Pairs of plants and turbines
n,N	Grid points for the relaxation of $hdn_{i,t}$
R	Grid points for supporting hyperplanes
$T_{t,\tau_{k,i}}$	Time periods to build wrap around constraints
UI	Turbines with identical features in the same plant

B. Parameters

$a_{i,l}$	Coefficients for the forebay level polynomials
$b_{i,l}$	Coefficients for the tailrace level polynomials
$D_{i,t,n}$	Grid points for the partition scheme for $d_{i,t}$
H_i	Water head [m]
$SC_{i,j}$	Start-up cost of turbine j in plant i [m.u.]
$P_{i,j}^{UP}$	Maximum power of turbine j in plant i [MW]
$Q_{i,j}^{U}$	Maximum outflow of turbine j in plant i [m^3/s]
$Q_{i,j,t,qq}$	Grid points for the partition scheme for $q_{i,j,t}$
V_i^{UP}	Target for the maximum storage of the reservoir of plant i at the end of the time horizon [Hm^3]
V_i^{LO}	Target for the minimum storage in plant i at the end of the time horizon [Hm^3]
$V0_i$	Initial storage of the reservoir of plant i [%]
VC	Conversion factor from [m^3/s] to [m^3/h]
WI_i	Forecast natural water inflow of plant i [m^3/s]
$\eta_{i,j}$	Average generation efficiency [MW/((m^3/s).m)]
λ_t	Forecast price of energy in period t [m.u./MWh]
ξ	Minimum water discharge factor
$\tau_{i,k}$	Time delay between plant i and plant k [h]
$\varphi_{i,j}$	Penstock head losses as a fraction of the net head
$\phi_{i,j}$	Constant, where $\phi_{i,j} = \eta_{i,j}\,(1-\varphi_{i,j})$

C. Variables

$c_{i,j,t}$	Start-up cost of unit j in plant i in period t [m.u.]
$d_{i,t}$	Total water discharge of plant i in period t [m^3/s]
$dn_{i,t,n}$	Disaggregated variable for $d_{i,t}$
$h_{i,t}$	Dummy variable defined as $h_{i,t} = hup_{i,t} - hdn_{i,t}$
$\underline{hdn}_{i,t}$	Tailrace level of plant i in period t [m]
$\overline{hdn}_{i,t}$	Overestimator variable for $hdn_{i,t}$
$hd_{i,j,t,qq}$	Convex-hull variable for water head [m]
$hup_{i,t}$	Forebay level of plant i in period t [m]
$h1_{i,j,t}$	Convex-hull variable for water head [m]
$h2_{i,j,t}$	Convex-hull variable for water head [m]
$p_{i,j,t}$	Power output of unit j in plant i in period t [MW]
$\overline{p}_{i,j,t}$	Dummy variable given by $\overline{p}_{i,j,t} = q_{i,j,t}h_{i,t}$
$ph_{i,t}$	Total power output of plant i in period t [MW]
$Profit$	Profit [m.u.]
$q_{i,j,t}$	Water discharge of unit j, plant i, period t [m^3/s]
$qd_{i,j,t,qq}$	Convex-hull variable for flow [m^3/s]
$qh_{i,t}$	Total water discharge in plant i in period t [m^3/s]
$s_{i,t}$	Spillage in plant i in period t [m^3/s]
$v_{i,t}$	Volume of the reservoir in plant i [Hm^3]
$x_{i,j,t}$	=1 if turbine j of plant i is on-line in period t, otherwise 0
$z_{i,j,t,qq}$	Convex-hull 0-1 variable to assign partition
$yd_{i,t,n}$	0-1 variable to assign partition

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I. Introduction

This work addresses the global optimization of head dependent Short Term Hydro Scheduling (STHS) problems using cascades of hydro plants. Mixed-Integer Linear Programming (MILP) models have received a good deal of attention in the literature for the STHS, due to the possibility of: 1) modeling several operating constraints such as ramp constraints or restricted operating regions; 2) using piecewise linear approximation submodels to replace nonlinearities of hydro power functions; and 3) using increasingly efficient MILP solvers such as GUROBI and CPLEX. A detailed MILP model from the point of view of operational constraints is proposed in [1]. These authors considered several operational constraints and used a simplified linear model to approximate the nonlinear hydro generation function, where the turbines within the plant are considered to have identical characteristics and are aggregated into one single unit. A more accurate piecewise linear sub-model was proposed in [2] to approximate the nonlinear relationship between the power output, the net water head, and the water discharged within the hydro generation function. This sub-model was further improved in [3], by including interpolations between the piecewise functions and by adopting a tighter continuous relaxation. NonLinear Programming (NLP) models have also been used to model the STHS [4], although these models restrict the utilization of startup costs or the enforcement of some operational constraints. The main sources of nonlinearities in STHS are the following: a) the hydro power generation function; b) the relation between the forebay level and the volume of water in the reservoir; and c) the relation between the tailrace level and the total water discharged by the plant. In general, the power generated by a turbine is given by the following equation [5]:

$$p_{i,j,t} = K\eta_{i,j}\zeta_{i,j}q_{i,j,t}H_{i,t} \quad \forall i,t,j, \tag{1}$$

where $p_{ij,t}$ denotes the output power of turbine j from plant i in the time period t, K is a constant, $\eta_{i,j}$ is the efficiency of the generator, $\zeta_{i,j}$ is the efficiency of the turbine, $q_{i,j,t}$ the flow discharged by turbine j and $H_{i,t}$ the net water head. Assuming that $\eta_{i,j}$ and $\zeta_{i,j}$ are constants as proposed in [6], (1) involves a bilinear term that is well known to give rise to nonconvexities that may lead NLP solvers to local solutions. The relations between the forebay and tailrace levels and the volume of water in the reservoir and the total water discharged by the plant, respectively, may be represented by linear functions, or by more complex polynomial functions [5], [6]. The level of detail used in the definition of these functions depends on the accuracy desired, but also on the turbines and topological characteristics of the plants. On the other hand, the utilization of complex functions is driven by the accuracy obtained, which in some cases may have a significant impact on the economic analysis of the systems.

Mixed-Integer NonLinear Programming (MINLP) models have been recently proposed for the STHS where detailed nonlinear expressions are considered [5]–[10]. The introduction of these nonlinearities, even using simplified equations as in [4], may lead to nonconvex MINLP models with multiple local optima, whereby MINLP solvers that rely on convexity may not guarantee global optimality of the solutions.

In terms of solution approaches the Lagrangian Relaxation (LR) is the most popular to solve large scale hydrothermal problems. The superior performance of LR is due to the decomposition of the original model into sub-problems, and the quality of the calculated bounds [11], [12]. A clear and concise review of the advantages and drawbacks of LR is given in [12]. LR has been also applied to the solution of the STHS problems [9], [13]–[15]. However, for a cascade of hydro plants the spatial-temporal interaction between the plants requires additional linking variables when compared with thermal systems. Additional decomposition algorithms involve for example the bi-level decompositions based on two levels of detail [8].

On the other hand, the direct solution of the STHS problem has relied on dynamic programming methods [16], [17], and on LP-based Branch & Bound (B&B) solvers for MILP problems [1]–[3], [11]. The recent trend on using nonlinearities in the STHS models, and their solution using MINLP solvers has led to the solution of nonconvex MINLP problems with solvers developed for convex MINLP problems [7], [10]. The current technology to solve MINLP problems is not as mature as the technology to solve MILP's. However, several algorithms are available [18]–[20], and implemented into commercial or open MINLP solvers such as DICOPT [21], α-ECP and SBB in GAMS, AAOA in AIMMS, MINOPT [22], and BONMIN [23]. Recently, significant advancements have also been made on the development of theory and algorithms for the deterministic global optimization of NLP and MINLP [24]–[27]. This is currently an active area of research in which there are currently available several solvers, such as BARON [28], LINDOGlobal [29], and Couenne [30], which can address the deterministic global solution of NLP and MINLP problems.

The objective of this work is to address the solution of STHS problems defined by MINLP models using a deterministic global optimization approach. The main contributions for this work are the following: 1) implementation of a spatial B&B algorithm to address the global optimization within a pre-specified tolerance of a detailed STHS MINLP model; 2) use of a specific type of constraints suggested by the STHS MINLP model formulation, namely Symmetry Breaking Constraints (SBC), applied to the binary variables associated with the status of the turbines; 3) a specific partition scheme for the relaxation of bilinear terms with semi-continuous variables; 4) comparison of the proposed approach for STHS models with a global optimization solver.

This work is motivated by the current trend to develop more detailed MINLP models for the STHS, and is supported by the following advancements: a) on global optimization ~~theory~~ for nonconvex MINLP problems; b) on the availability of affordable multiple threads computer hardware; and c) on the increasing sophistication of MILP solvers as a result of the implementation of new cuts based on polyhedral theory, as well as to the inclusion of heuristics and meta-heuristics within the MILP solvers that help finding integer solutions [31]–[33].

II. Problem statement

Given is a set of hydro plants in cascade that produce electricity for the day-ahead market. Some plants can store water in a reservoir, while others are run-of-the-river plants. Each plant has a set of turbines with a maximum output flow and power generated, linked to the same reservoir. The problem is to determine the start and duration of operation of each turbine, and the respective power output that maximize the operating profit, subject to the limits of the reservoirs, the mass balances of water, and the operating limits of the turbines. Each reservoir has as inputs a deterministic natural inflow and the discharge from upstream plants, and as outputs the flow discharged by each turbine linked to the reservoir. The profit is calculated as the difference between the revenues of selling electricity minus the start-up costs of the turbines. The value of the water is not considered in the profit, since the volume of each reservoir at the end of the time horizon must be greater than or equal to the initial volume. The system is considered as an electricity price taker, with the price of electricity following a given hourly profile. The time horizon is equal to one day, discretized in periods of one hour. For each pair of plants (i, i') there is a time delay between the total flow discharged from plant i to plant i'. The system does not have to match a specific demand pattern, since all energy produced is delivered, without considering electrical network constraints.

III. MINLP MODEL

In this work a MINLP model is proposed for STHS based on the specific hydro parts of the test cases described in [6]. The mathematical formulation is the following:

$$\text{Maximize } Profit = \sum_i \sum_t \lambda_t ph_{i,t} - \sum_i \sum_j \sum_t c_{i,j,t}, \quad (2)$$

subject to:

$$c_{i,j,t} \geq SC_{i,j}\left(x_{i,j,t} - x_{i,j,t-1}\right) \quad \forall i, j \in M, t > 1, \quad (3)$$

$$c_{i,j,t} \geq SC_{i,j}\left(x_{i,j,t} - x_{i,j,tt}\right) \quad \forall i, j \in M, t = 1, tt = 24, \quad (4)$$

$$hup_{i,t} = a_{i,0}+a_{i,1}v_{i,t}+a_{i,2}v_{i,t}^2+a_{i,3}v_{i,t}^3+a_{i,4}v_{i,t}^4 \quad \forall i, t, \quad (5)$$

$$hdn_{i,t} = b_{i,0}+b_{i,1}d_{i,t}+b_{i,2}d_{i,t}^2+b_{i,3}d_{i,t}^3+b_{i,4}d_{i,t}^4 \quad \forall i, t, \quad (6)$$

$$d_{i,t} = qh_{i,t} + s_{i,t} \quad \forall i, t, \quad (7)$$

$$qh_{i,t} = \sum_{j \in M} q_{i,j,t} \quad \forall i, t, \quad (8)$$

$$v_{i,t} = v_{i,t-1} + VC(W_i - d_{i,t} + \sum_{k \in IC_{k,i}} \sum_{tt \in T_{t,\tau_{k,i}}} d_{k,tt}) \, \forall i, t, \quad (9)$$

$$p_{i,j,t} = \phi_{i,j} q_{i,j,t}(hup_{i,t} - hdn_{i,t}) \quad \forall i, j \in M, t, \quad (10)$$

$$ph_{i,t} = \sum_{j \in M} p_{i,j,t} \quad \forall i, t, \quad (11)$$

$$\xi Q_{i,j}^U x_{i,j,t} \leq q_{i,j,t} \leq Q_{i,j}^U x_{i,j,t} \quad \forall i, j \in M, t, \quad (12)$$

$$p_{i,j,t} \leq P_{i,j}^{UP} x_{i,j,t} \quad \forall i, j \in M, t, \quad (13)$$

$$v_{i,t} \geq V0_i \quad \forall i, t = 24. \quad (14)$$

$$c_{i,j,t}, hup_{i,t}, hdn_{i,t}, v_{i,t}, d_{i,t}, s_{i,t}, p_{i,j,t} \geq 0, x_{i,j,t} \in \{0, 1\} \quad \forall i, j \in M, t, \quad (15)$$

The main features of this model are the following: a) the utilization of a nonlinear function for the calculation of the power generated, represented by (10); and b) the use of polynomial functions to calculate the forebay and tailrace levels, see constraints (5) and (6). In the current model it is also assumed that each plant may have multiple turbines linked to the same reservoir. Therefore, binary variables are used to account for their startup costs and to enforce ranges of operation. Note that with some exceptions, for example [8], [9], there are few works in the literature that address multiple turbines per hydro plant.

A. Simplified MINLP model

The proposed MINLP model can be simplified by neglecting in (5) and (6) the terms of the polynomial function of order greater than one, leading to the linear relationships:

$$hup_{i,t} = a_{i,0} + a_{i,1} v_{i,t} \quad \forall i, t, \quad (16)$$

$$hdn_{i,t} = b_{i,0} + b_{i,1} d_{i,t} \quad \forall i, t. \quad (17)$$

This is a rough but straightforward approximation to the original polynomial functions, which is used here with the objective of comparing the computational performance of MINLP models with and without polynomial functions. Note that better approximations of the equations may be obtained by adjusting the coefficients of the linear equation with the original data. However, this is out of the scope of this paper. Therefore, a simplified model (S-MINLP) is proposed by replacing (5) and (6) with (16) and (17) in the MINLP model.

B. MILP model

In this section an alternative model is proposed by further simplifying the power generation function. Here, the variations on the net water head are neglected, i.e. the difference between the forebay and tailrace levels is assumed to be constant, and denoted by H_i. H_i is defined as the difference between the average forebay level calculated for the lower and upper values of the variable $v_{i,t}$, and the average tailrace level calculated for the lower and upper values of $d_{i,t}$. With these assumptions the power generated is given by the following equation:

$$p_{i,j,t} = \phi_{i,j} q_{i,j,t} H_i \quad \forall i, j \in M, t. \quad (18)$$

In this case, the simplified MILP (S-MILP) model does not involve bilinear terms nor the polynomial functions. This means that from the original MINLP, (5), (6), and (10) are not considered and (18) is added.

IV. SOLUTION APPROACH

In order to find the global optimum within a pre-specified tolerance, a spatial Hydro Branch & Bound (sHBB) framework is proposed, and tailored for the STHS. This approach is based on a branch and bound search, where on each node an MILP Overestimator Problem (MILP-OEP) and an MINLP model are solved to obtain an upper and lower bound on the profit, respectively. The MILP-OEP provides a tight linear overestimation of the nonconvex region of the original MINLP problem, and thus a valid upper bound of the profit on each node of sHBB tree. The MILP-OEP is built over the relaxation of the nonlinearities, whereby the bilinear terms and polynomial functions are replaced by polyhedral envelopes that overestimate the feasible region of the original problem.

A. Relaxation of the bilinear terms

In the MILP-OEP the bilinear terms are replaced by a specific dynamic piecewise linear estimator model, which defines the convex hull envelopes over partitions of the range of the variables involved. In (10) the variables involved are $q_{i,j,t}$, $hup_{i,t}$ and $hdn_{i,t}$, which can be manipulated defining a new variable $h_{i,t} = hup_{i,t} - hdn_{i,t}\,\forall i,t$, while the bilinear terms to be tackled are defined as $\overline{p}_{i,j,t} = q_{i,j,t}h_{i,t}\,\forall i,j \in M,t$, which leads to the following linearized equation for the hydro power generation function used in the MILP-OEP:

$$p_{i,j,t} = \phi_{i,j}\overline{p}_{i,j,t} \quad \forall i,j \in M,t. \tag{19}$$

The relaxation of the bilinear terms are built employing the convex envelopes proposed in [34], whereby in the MILP-OEP, $\overline{p}_{i,j,t} = q_{i,j,t}h_{i,t}$ is replaced by the following inequalities, the well known McCormick inequalities,

$$\overline{p}_{i,j,t} \geq Q^L_{i,j,t}h_{i,t} + q_{i,j,t}H^L_{i,t} - Q^L_{i,j,t}H^L_{i,t} \quad \forall i,j \in M,t, \tag{20}$$

$$\overline{p}_{i,j,t} \geq Q^U_{i,j,t}h_{i,t} + q_{i,j,t}H^U_{i,t} - Q^U_{i,j,t}H^U_{i,t} \quad \forall i,j \in M,t, \tag{21}$$

$$\overline{p}_{i,j,t} \leq Q^U_{i,j,t}h_{i,t} + q_{i,j,t}H^L_{i,t} - Q^U_{i,j,t}H^L_{i,t} \quad \forall i,j \in M,t, \tag{22}$$

$$\overline{p}_{i,j,t} \leq Q^L_{i,j,t}h_{i,t} + q_{i,j,t}H^U_{i,t} - Q^L_{i,j,t}H^U_{i,t} \quad \forall i,j \in M,t. \tag{23}$$

Note that the variable $q_{i,j,t}$ is a semi-continuous variable defined as $q_{i,j,t} \in \{0\} \cup [Q^L_{i,j,t}, Q^U_{i,j,t}]$, which in this work is taken into account when the above convex envelopes are built. This provides a tighter relaxation of the bilinear terms, which is not taken into account for example by the solver BARON. The error introduced by the relaxation, defined as $|\overline{p}_{i,j,t} - q_{i,j,t}h_{i,t}|$, may be reduced by considering the convex envelopes built over a partition of the domain of the variables $q_{i,j,t}$ as suggested in [26]. The formal definition of the convex envelopes built over a partition scheme with grid points $qq \in G$ is defined by the following disjunction:

$$\bigvee_{qq \in G} \begin{bmatrix} Z_{i,j,t,qq} \\ \overline{p}_{i,j,t} \geq Q_{i,j,t,qq-1}h_{i,t} + q_{i,j,t}H^L_{i,t} - Q_{i,j,t,qq-1}H^L_{i,t} \\ \overline{p}_{i,j,t} \geq Q_{i,j,t,qq}h_{i,t} + q_{i,j,t}H^U_{i,t} - Q_{i,j,t,qq}H^U_{i,t} \\ \overline{p}_{i,j,t} \leq Q_{i,j,t,qq}h_{i,t} + q_{i,j,t}H^L_{i,t} - Q_{i,j,t,qq}H^L_{i,t} \\ \overline{p}_{i,j,t} \leq Q_{i,j,t,qq-1}h_{i,t} + q_{i,j,t}H^U_{i,t} - Q_{i,j,t,qq-1}H^U_{i,t} \end{bmatrix} \quad \forall i,j \in M,t, \tag{24}$$

where $Z_{i,j,t,qq}$ is a boolean variable. This disjunction can be re-written as a sub-MILP model by using a convex-hull reformulation [35]:

$$\overline{p}_{i,j,t} \geq \sum_{qq \in G} \big(Q_{i,j,t,qq-1}hd_{i,j,t,qq-1} + qd_{i,j,t,qq-1}H^L_{i,t} - Q_{i,j,t,qq-1}H^L_{i,t}z_{i,j,t,qq}\big) \quad \forall i,j \in M,t, \tag{25}$$

$$\overline{p}_{i,j,t} \geq \sum_{qq \in G} \big(Q_{i,j,t,qq}hd_{i,j,t,qq-1} + qd_{i,j,t,qq-1}H^U_{i,t} - Q_{i,j,t,qq}H^U_{i,t}z_{i,j,t,qq}\big) \quad \forall i,j \in M,t, \tag{26}$$

$$\overline{p}_{i,j,t} \leq \sum_{qq \in G} \big(Q_{i,j,t,qq}hd_{i,j,t,qq-1} + qd_{i,j,t,qq-1}H^L_{i,t} - Q_{i,j,t,qq}H^L_{i,t}z_{i,j,t,qq}\big) \quad \forall i,j \in M,t, \tag{27}$$

$$\overline{p}_{i,j,t} \leq \sum_{qq \in G} \big(Q_{i,j,t,qq-1}hd_{i,j,t,qq-1} + qd_{i,j,t,qq-1}H^U_{i,t} - Q_{i,j,t,qq-1}H^U_{i,t}z_{i,j,t,qq}\big) \quad \forall i,j \in M,t, \tag{28}$$

$$q_{i,j,t} = \sum_{qq \in G} qd_{i,j,t,qq-1} \quad \forall i,j \in M,t \tag{29}$$

$$h1_{i,j,t} = \sum_{qq \in G} hd_{i,j,t,qq-1} \quad \forall i,j \in M,t, \tag{30}$$

$$h2_{i,j,t} \leq H^U_{i,t}z_{i,j,t,qq} \quad \forall i,j \in M, qq = 1,t, \tag{31}$$

$$h_{i,t} = h1_{i,j,t} + h2_{i,j,t} \quad \forall i,j \in M,t, \tag{32}$$

$$qd_{i,j,t,qq-1} \geq Q_{i,j,t,qq-1}z_{i,j,t,qq}\,\forall i,j \in M,t,qq \in G, \tag{33}$$

$$qd_{i,j,t,qq-1} \leq Q_{i,j,t,qq}z_{i,j,t,qq} \quad \forall i,j \in M,t,qq \in G, \tag{34}$$

$$H^L_{i,t}z_{i,j,t,qq} \leq hd_{i,j,t,qq-1} \leq H^U_{i,t}z_{i,j,t,qq} \quad \forall i,j \in M,t,qq \in G, \tag{35}$$

$$\sum_{qq>1} z_{i,j,t,qq} = 1 \quad \forall i,j \in M,t, \tag{36}$$

Increasing the number of partitions reduces the gap between $\overline{p}_{i,j,t}$ and $q_{i,j,t}h_{i,t}$, but also increases the number of equations and continuous and binary variables, and consequently the computational time associated with the solution of the MILP-OEP. In the context of an hydrothermal model, Cerisola et al. [36] have recently proposed the same type of linear approximation using piecewise McCormick planes. However, they use a Big-M reformulation, instead of the convex-hull. Preliminary results have shown us that in the problems studied in this work the convex-hull provides a tighter relaxation than the Big-M reformulation.

B. Relaxation of the polynomial functions

In this work three polynomial relaxations (PR) are considered for building linear envelopes for the polynomial functions: 1) PR1 - based on the determination of the inflection points of the polynomials that are nonconvex and nonconcave and calculation of off-set values to be included in the linear over and under estimators functions; 2) PR2 - built by replacing each univariate nonlinear power function of the polynomial function by a new variable, and then for each univariate nonlinear power function an overestimation model is built; and 3) PR3 - equal to PR2 but only built over the nonconvex and nonconcave functions, while for the remaining polynomial functions an overestimation is built over the original function. The three relaxations are rigorous in the sense that the linearizations do not cut-off any part of the polynomial functions. However, PR1 and PR3 require a preprocessing step to identify the characteristics of the polynomial functions, while PR2 does not require that step.

An analysis of the properties of the polynomial functions was made in order to determine the sign of the second derivative through the identification of the inflection points, and hence the polynomial functions that are concave, convex, and nonconvex and nonconcave. Valid over and under estimators for the concave and convex polynomial functions are built using piecewise linear approximations between the bounds of the variables and hyperplanes at given points. Due to space limitations, only the construction of these estimators are presented for the concave polynomials associated with the tailrace level. The under-estimators are built over a partition N with grid points n, where the over/under estimator value is represented by $\overline{hdn}_{i,t}$. In order to simplify the equations, $\Theta(D_{i,t,n})$ is defined first as,

$$\Theta(D_{i,t,n}) = b_{i,0} + b_{i,1}D_{i,t,n} + b_{i,2}D_{i,t,n}^2 + b_{i,3}D_{i,t,n}^3 + b_{i,4}D_{i,t,n}^4 \quad \forall i,t. \tag{37}$$

The piecewise underestimation is as follows:

$$\overline{hdn}_{i,t} \geq \sum_{n\in N}\left[\Theta(D_{i,t,n})yd_{i,t,n} + \frac{\Theta(D_{i,t,n+1}) - \Theta(D_{i,t,n})}{D_{i,t,n+1} - D_{i,t,n}} (dn_{i,t,n} - D_{i,t,n}yd_{i,t,n})\right] \quad \forall i,t, \tag{38}$$

$$d_{i,t} = \sum_{n\in N} dn_{i,t,n} \quad \forall i,t, \tag{39}$$

$$D_{i,t,n}yd_{i,t,n} \leq dn_{i,t,n} \leq D_{i,t,n+1}yd_{i,t,n} \, \forall i,t,n \in N, \tag{40}$$

$$\sum_{n\in N} yd_{i,t,n} = 1 \quad \forall i,t. \tag{41}$$

The over estimators are built using the supporting hyperplanes:

$$\overline{hdn}_{i,t} \leq \Theta(D_{i,t,r}) + \nabla\Theta(D_{i,t,r})(d_{i,t} - D_{i,t,r}) \quad \forall i,t,r. \tag{42}$$

An equivalent rationale is employed to build under and over estimators for the concave polynomial functions for the forebay levels. As discussed in [12] on the applicability of piecewise linear functions for hydrothermal models, these functions provide an approximation of the real function. However, in this work, they are used to predict bounds.

C. Specific details and remarks

The length of the interval of the variables is known to have an impact on the tightness of the relaxation provided by MILP-OEP. Therefore, it is important to eliminate infeasible regions out of the domain of the variables. In this work, a preprocessing step is performed in order to contract the bounds of the variables by solving two LP problems for each variable, whereby a variable is minimized/maximized subject to the constraints of the MILP-OEP. Through this procedure, the lower and upper bounds of the variables may be tightened. The formulation of the MILP-OEP is improved by enforcing symmetry breaking constraints over the binary variables $x_{i,j,t}$ for the turbines with identical specifications that operate in parallel and are linked to the same reservoir. These constraints are represented by:

$$x_{i,j,t} \geq x_{i,j+1,t} \quad \forall i,j \in M \cap UI_{i,j}, i, j+1 \in M, t, \tag{43}$$

where $UI_{i,j}$ represents the subsets of turbines with the same characteristics in the same plant. The sHBB algorithm solves an MILP-OEP and an MINLP model at each node of the tree. The branching process involves splitting the feasible region of the MILP-OEP based on the largest error of the relaxations for the bilinear terms and for the polynomial functions. If this error is associated with a bilinear term, the respective semi-continuous variable $q_{i,j,t}$ is split into two regions in the middle point, $q'_{i,j,t}$, of the interval $[Q^L_{i,j,t}, Q^U_{i,j,t}]$, and two new problems are generated, leading to two new nodes. On each node the following procedure is applied: 1) the bounds of $q_{i,j,t}$ are updated, and consequently the grid used in the relaxation is updated; 2) the SBC induces an additional bound contraction scheme, which is applied to the turbines with the same characteristics of the turbine selected to make the branch. If in one of the generated nodes, the upper bound is changed, then the upper bounds of the turbines for $j' > j$ are also updated:

$$Q^U_{i,j,t} \geq Q^u_{i,j+1,t} \quad \forall i,j \in M \cap UI_{i,j}, i, j+1 \in M, t. \tag{44}$$

Similarly, for the node where the lower bound is updated, the following bounds are enforced:

$$Q^L_{i,j+1,t} \geq Q^U_{i,j,t} \quad \forall i,j \in M \cap UI_{i,j}, i, j+1 \in M, t. \tag{45}$$

3) If the MILP-OEP is solved to optimality within the maximum CPU time set to solve the MILP-OEP, one additional partition is added to the piecewise partition scheme of all variables $q_{i,j,t'}$ $\forall t'$ in the two new generated nodes; otherwise the number of partitions of the two new generated nodes is set to the number of partitions of the precedent node, and the number of partitions is not increased any further. Other details of the sHBB implementation include: 1) the selection of the next node to solve is made based on the node with the largest upper bound; 2) the MILP-OEPs are solved within a specified CPU time limit. If the problem is solved to optimality, then the upper bound in the node is given by the integer solution; otherwise, the best bound obtained is used as a valid upper bound in the node; and 3) the lower bound of the original MINLP problem is obtained by fixing the binary variables associated with the turbines at the value of the MILP-OEP solution, if they are equal to one, leaving the variables equal to zero free and solving a reduced MINLP problem. This MINLP problem provides a valid lower bound on the objective function at a low computational cost since the number of free binary variables is reduced. This approach proved to be a better option than to fix all the binary variables and solve an NLP problem.

V. Computational experiments

The computational performance of the proposed models and of the global optimization approach are evaluated in this section. We considered four test cases presented in [6] involving cascades of hydro plants with different number of plants, turbines and cascade topology: Case 1 - 4 plants, 24 turbines; Case 2 - 5 plants, 22 turbines; Case 3 - 7 plants, 29 turbines; Case 4 - 6 plants, 44 turbines. The configuration of the cascades and the topological, reservoir and hydro data are the same as those published in [6]. The size of the four models is presented in Appendix A. The models and the global optimization approach are implemented in GAMS [37] and

TABLE I
COMPARISON BETWEEN sHBB AND BARON FOR THE DETAILED MINLP.

	sHBB						BARON		
	PR1			PR2					
Case	P (m.u.)	G (%)	T (s)	P (m.u.)	G (%)	T (s)	P (m.u.)	G (%)	T (s)
Case 1	1,249,705	0.19	404	1,249,705	0.20	52	1,249,243	0.50	106
Case 2	1,372,403	0.15	25	1,372,403	0.22	32	1,371,840	2.67	10,800
Case 3	2,552,090	0.22	71	2,551,783	0.38	65	-426,000	-	10,800
Case 4	2,847,564	2.68	10,800	2,847,392	3.44	10,800	2,845,660	6.04	10,800

P - Profit, G - Gap, T - CPU time, PR1, PR2 - Polynomial relaxations.The terminal criteria are set to 0.5% gap and to a maximum CPU time of 10,800s.

solved on a computer with an Intel Core i7@3.07GHz CPU, 64 bits, and 8Gb of RAM. The solvers used are CPLEX 12.4, GAMS/DICOPT 23.8 and BARON 11.1. Table I summarizes the computational performance of sHBB and BARON for Cases 1, 2, 3 and 4 for the detailed MINLP (the indicated CPU time set for sHBB preclude the time spent on bound tightening, which was respectively 21, 15, 23 and 31 min). The terminal criteria are set to 0.5% gap and to a maximum CPU time of 10,800s. For sHBB, the table presents the results with the polynomial relaxations PR1 and PR2, showing that optimality gaps below 0.5% gap are obtained in Cases 1, 2 and 3, in short CPU times i.e. all below 404s and 65s for PR1 and PR2, respectively. However, for the larger Case 4, the performance decreases and solutions with an optimality gap below 0.5% are not reached within 10,800s. The computational performance of PR3 is not presented due to space limitations, but it is between PR1 and PR2. The results obtained with BARON, show that for Case 3 it reveals difficulty in finding a positive lower bound, and that comparing BARON with the proposed approach, sHBB is found to achieve a greater or at least equal lower bound for the four cases. BARON is only able to solve Case 1 within the specified optimality gap in 106s, while for Cases 2, 3, and 4 it cannot reach solutions within 0.5% optimality gap within 10,800s, which highlights the performance of sHBB.

The results obtained with the S-MINLP with sHBB and BARON present the same trend as obtained with the detailed MINLP, but in general with shorter CPU times and smaller optimality gaps for sHBB, see Table II. Comparing

TABLE II
COMPARISON BETWEEN sHBB AND BARON FOR THE S-MINLP. THE TERMINAL CRITERIA ARE SET TO 0.5% GAP AND TO A MAXIMUM CPU TIME OF 10,800S.

	sHBB			BARON		
Case	P (m.u.)	G (%)	T (s)	P (m.u.)	G (%)	T (s)
Case 1	1,303,536	0.16	51	1,303,766	0.50	56
Case 2	1,447,638	0.14	16	-312,432	-	10,800
Case 3	2,699,187	0.21	39	2,681,850	2.18	10,800
Case 4	2,907,365	2.32	10800	2,906,790	3.48	10,800

P - Profit, G - Gap, T - CPU time.

the results of the MINLP model with the approximations for the polynomial functions, and without them, S-MINLP, it is clear that the bilinear terms have a major influence on the observed gaps between the lower and upper bounds of the objective function. This is supported by the small differences between the performance of sHBB and BARON with the MINLP and S-MINLP models. In order to assess the objective function values obtained with sHBB, a local solver for MINLP problems is used to solve the detailed MINLP and S-MINLP models. The two models are solved either per se or including the SBC, in order to check the effect of these constraints. Table III shows the Symmetry Breaking Constraints (SBC) to have an all-round strong impact on the performance of DICOPT. This is explained by the symmetry breaking imposed by these constraints during the solution of the master problem within DICOPT. Within sHBB these constraints have also an important role on the solution of the MILP-OEP problem and on the solution of the MINLP problem used to calculate the lower bound in each node.

TABLE III
MINLP MODELS SOLVED WITH DICOPT, AND S-MILP WITH CPLEX.

	Without SBC		With SBC	
Cases	P (m.u.)	T (s)	P (m.u.)	T (s)
MINLP				
Case 1	1,249,705	9	1,249,705	9
Case 2	1,364,140	10,800	1,369,935	39
Case 3	2,545,173	10,800	2,551,687	8,934
Case 4	2,846,776	10,804	2,848,254	10,654
S-MINLP				
Case 1	1,303,782	5	1,303,782	6
Case 2	1,441,263	10,800	1,436,125	43
Case 3	2,685,520	10,800	2,690,461	291
Case 4	2,906,996	10,803	2,906,496	7,422
S-MILP				
Case 1	1,253,778	0.1	1,253,778	0.1
Case 2	1,363,801	1.1	1,363,801	0.2
Case 3	2,546,971	11.8	2,546,971	1.4
Case 4	2,867,973	31.9	2,867,973	2.4

P - Profit, T - CPU time.

A. Scheduling results

Analyzing the daily profit for Case 2 of MINLP-PR1 found with the proposed approach, i.e. 1,372,403m.u., it presents a higher profit of 2,468m.u. compared to DICOPT with SBC (1,369,935m.u.) which yields to an additional yearly profit of 900,820m.u. Similar analysis can be made for the other cases resulting always in a positive gain by using the sHBB. The exception is Case 4, where DICOPT obtained a better profit, most likely as a result of the implemented integer cuts forcing the algorithm to fix a different combination of binary variables on each iteration, which are missing in sHBB. As for the S-MILP model, the CPU times required for solving it with CPLEX are significantly lower than the ones obtained with the global optimization approach. Fig. 1, 2 and 3 illustrate the power production schedule for Case 2 obtained with sHBB, BARON and the S-MILP model. These schedules have the same general ~~trend, however~~ a thorough analysis shows differences on the turbines activated and respective water flows. As an example, in the periods 4, 5 and 6, the S-MILP does not activate the turbines from plant H6, which activate in the remaining two schedules. As a consequence, different operating conditions in terms of the profile of the volume of water in the reservoir H6 and the spillage of the downstream run of the river plant H7 are obtained with each model (see Fig. 4 and 5). The analysis and comparison of the results obtained should take into consideration that the models have different levels of accuracy and constraints. For example, the detailed MINLP relates the forebay and tailrace levels, respectively, with the volume and water discharged, leading to a variable net water head, while for the S-MILP it is constant. This feature alone has a major impact on the power generated and consequently on the profit obtained.

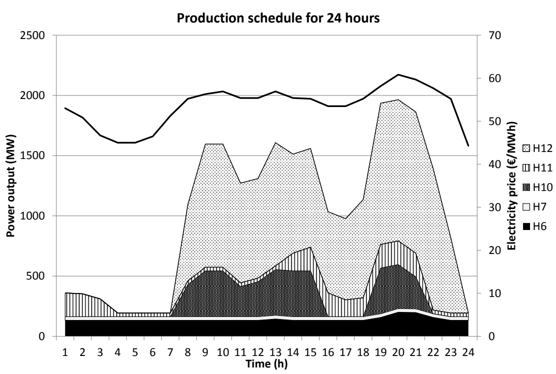

Fig. 1. Case 2 production schedule obtained with sHBB.

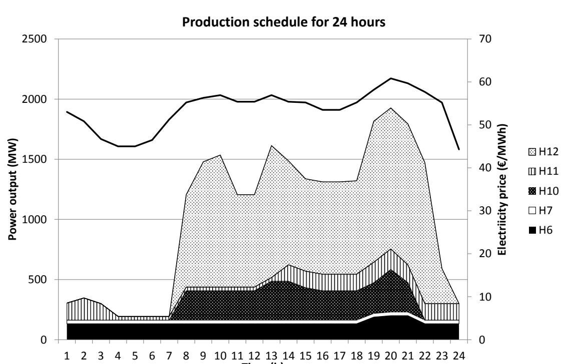

Fig. 2. Case 2 production schedule obtained with BARON.

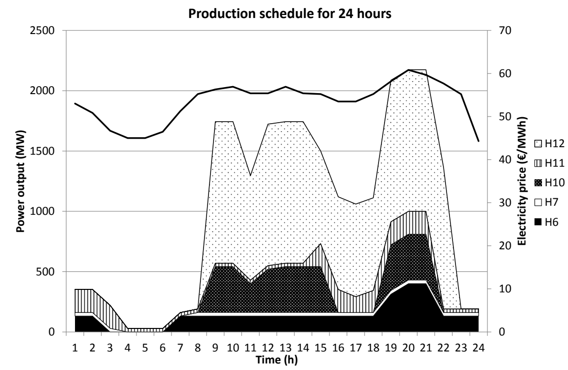

Fig. 3. Case 2 production schedule obtained with S-MILP.

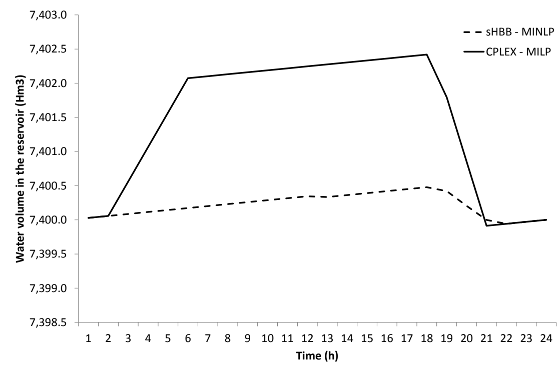

Fig. 4. Case 2 water volumes of reservoir H6 obtained with sHBB and MINLP and CPLEX and s-MILP.

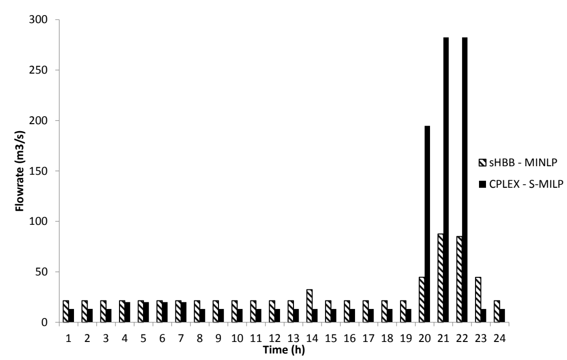

Fig. 5. Case 2 spillage of run of the river plant H7 obtained with sHBB and MINLP and CPLEX and s-MILP.

VI. Conclusions

In this paper a tailored global optimization algorithm, sHBB, is proposed for the STHS of a cascade of hydro plants. A nonconvex MINLP model for the operation of the cascade is formulated and an overestimator model based on a relaxation framework for: a) the polynomial functions describing the forebay and tailrace levels; ~~and~~ b) the bilinear terms in the power generation function. The overall performance of the proposed approach is the result of the combination of several algorithmic implementations, such as a specific dynamic partition relaxation of the bilinear terms exploiting the semi-continuous characteristics of one of the variables, and of the symmetry breaking constraints that were shown to play an important role in the solution of the MILP overestimator problem. One relevant conclusion achieved is the establishment of the bilinear terms as the nonlinear functions with a major contribution for the gap between the lower and the upper bound with sHBB. This assessment arises from comparing the approximations errors and final gaps obtained with sHBB for the detailed MINLP and simplified model, S-MINLP. Another important result obtained with sHBB is the identification of alternative power production schedules, which come associated with higher profits, a possibility worth of further investigation. The proposed framework was found to exhibit better computational performance than a current available standard global optimization solver and provide better solutions than a local MINLP solver, making it potentially suitable to build relaxation models for nonconvex hydrothermal models in diverse optimization frameworks. The algorithmic implementations proposed are a contribution for the application of deterministic global optimization in power systems.

APPENDIX A

TABLE IV
SIZE OF THE MODELS, MINLP/MILP-OEP/S-MILP-OEP

Cases	Equations	Variables	0-1 Variables
Case 1	3,889/22,321/20,017	2,977/14,185/13,081	576/4,200/3,720
Case 2	3,697/13,825/11,209	2,953/ 7,921/ 6,769	528/2,064/1,584
Case 3	4,945/18,673/14,857	3,961/10,729/ 9,001	696/2,808/2,088
Case 4	6,937/24,049/21,865	5,233/13,513/12,769	1,056/3,408/3,168

MILP-OEP - Overestimation model of the original MINLP model with PR1 at the root node. S-MILP-OEP - Overestimation model of the S-MINLP model at the root node.

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