Polyhedral Approach
to
Integer Linear Programming

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Brief history

First Algorithms | Polynomial Algorithms

Solving systems of linear equations

- Babylonians 1700BC
- Gauss 1801

- Edmonds 1967

Solving systems of linear inequalities

- Fourier 1822
- Dantzig 1951

- Khachyan 1979
- Karmarkar 1984

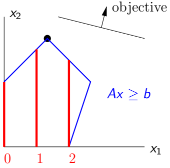
Solving systems of linear inequalities in integers

- Gomory 1958

- Lenstra 1983

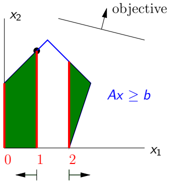
Mixed Integer Linear Programming

$$\begin{array}{c} \min cx \\ x \in S \end{array} \qquad \text{where } S := \{x \in \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p} : \ Ax \geq b\}$$



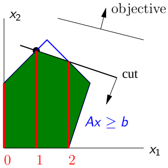
Linear Relaxation

$$\begin{array}{c} \min cx \\ x \in P \end{array} \qquad \text{where } P := \{x \in \mathbb{R}_+^n : \ Ax \geq b\}$$



Branch-and-bound

Land and Doig 1960



Cutting Planes

Dantzig, Fulkerson and Johnson 1954

Gomory 1958

Polyhedral Theory

$P := \{x \in \mathbb{R}^n_+ : \ Ax \geq b\}$ Polyhedron

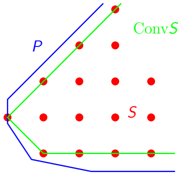
$S := P \cap (\mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+)$ Mixed Integer Linear Set

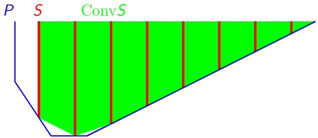
$$\text{Conv } S := \{x \in \mathbb{R}^n : \ \exists x^1, \ldots, x^k \in S, \ \lambda \geq 0, \ \sum \lambda_i = 1 \ \text{such that } x = \lambda_1 x^1 + \ldots + \lambda_k x^k\}$$

THEOREM Meyer 1974

If A, b have rational entries, then Conv S is a polyhedron.

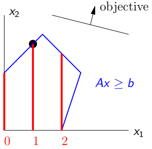
Proof Using a theorem of Minkowski 1896 and Weyl 1935 :
P is a polyhedron if and only if $P = Q + C$ where Q is a polytope and C is a polyhedral cone.





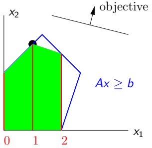
Thus

$$\begin{aligned} \min \quad & cx \\ & x \in S \end{aligned}$$



can be rewritten as the LP

$$\begin{aligned} \min \quad & cx \\ & x \in \text{Conv } S \end{aligned}$$



We are interested in the constructive aspects of Conv S.

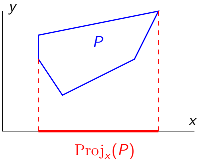
REMARK The number of constraints of Conv S can be exponential in the size of $Ax \geq b$, BUT
1) sometimes a partial representation of Conv S suffices (Example : Dantzig, Fulkerson, Johnson 1954) ;
2) Conv S can sometimes be obtained as the projection of a polyhedron with a polynomial number of variables and constraints.

Projections

Let $P := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k : \; Ax + Gy \geq b\}$

DEFINITION

$\text{Proj}_x(P) := \{x \in \mathbb{R}^n : \; \exists y \in \mathbb{R}^k \text{ such that } Ax + Gy \geq b\}$



THEOREM

$\text{Proj}_x(P) = \{x \in \mathbb{R}^n : \; vAx \geq vb \text{ for all } v \in Q\}$
where $Q := \{v \in \mathbb{R}^m : \; vG = 0, v \geq 0\}$.

PROOF

Let $x \in \mathbb{R}^n$. Farkas's lemma (Farkas 1894) implies that
$Gy \geq b - Ax$ has a solution y if and only if
$v(b - Ax) \leq 0$ for all $v \geq 0$ such that $vG = 0$. ■

Fractional Cuts   Gomory 1958

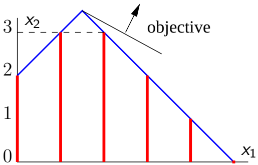
Consider a single constraint : $S := \{x \in \mathbb{Z}^n_+ : \sum_{j=1}^n a_j x_j = b\}$.

Let $b = \lfloor b \rfloor + f_0$ where $0 < f_0 < 1$,
and $a_j = \lfloor a_j \rfloor + f_j$ where $0 \le f_j < 1$.

THEOREM $\sum_j f_j x_j \ge f_0$ is a valid inequality for S.

EQUIVALENT FORM $\sum_j \lfloor a_j \rfloor x_j \le \lfloor b \rfloor$.

APPLICATION



$$\begin{aligned} \max\ z = x_1 + 2x_2 \\ -x_1 + x_2 \le 2 \\ x_1 + x_2 \le 5 \\ x_1 \in \mathbb{Z}_+ \\ x_2 \in \mathbb{R}_+ \end{aligned}$$

$$\begin{aligned} z + 0.5x_3 + 1.5x_4 &= 8.5 \\ x_1 - 0.5x_3 + 0.5x_4 &= 1.5 \\ x_2 + 0.5x_3 + 0.5x_4 &= 3.5 \end{aligned}$$

Cut $0.5x_3 + 0.5x_4 \ge 0.5$
or $x_2 \le 3$.

Mixed Integer Cuts Gomory 1963

Consider a single constraint : $S := \{x \in \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p} : \sum_{j=1}^n a_j x_j = b\}$.

Let $b = \lfloor b \rfloor + f_0$ where $0 < f_0 < 1$,
and $a_j = \lfloor a_j \rfloor + f_j$ where $0 \leq f_j < 1$.

THEOREM

$$\sum_{\substack{j \leq p: \\ f_j \leq f_0}} \frac{f_j}{f_0} x_j + \sum_{\substack{j \leq p: \\ f_j > f_0}} \frac{1 - f_j}{1 - f_0} x_j + \sum_{\substack{j \geq p+1: \\ a_j > 0}} \frac{a_j}{f_0} x_j - \sum_{\substack{j \geq p+1: \\ a_j < 0}} \frac{a_j}{1 - f_0} x_j \geq 1$$

is a valid inequality for S.

NOTE The mixed integer cuts dominate the fractional cuts.

Experiments of Bonami and Minoux 2005 on MIPLIB 3 instances give the amount of duality gap $= \min_{x \in S} cx - \min_{x \in P} cx$ closed by strengthening P with mixed integer cuts from the optimal basis :

gap closed : 24 %

Yet, for thirty years, fractional cuts and mixed integer cuts were not used in MILP solvers.

In 1991, Gomory remembered his experience with fractional cuts as follows : In the summer of 1959, I joined IBM research and was able to compute in earnest... We started to experience the unpredictability of the computational results rather steadily.

In 1991, Padberg and Rinaldi made the following comments : These cutting planes have poor convergence properties... classical cutting planes furnish weak cuts... A marriage of classical cutting planes and tree search is out of the question as far as the solution of large-scale combinatorial optimization problems is concerned.

In 1989, Nemhauser and Wolsey had this to say : They do not work well in practice. They fail because an extremely large number of these cuts frequently are required for convergence.

In 1985, Williams says : Although cutting plane methods may appear mathematically elegant, they have not proved very successful on large problems.

In 1988, Parker and Rardin give the following explanation for this lack of success : The main difficulty has come, not from the number of iterations, but from numerical errors in computer arithmetic.

GOMORY CLOSURE

$Ax \geq b \quad x \in \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p}$

- Every valid inequality for $P := \{x \geq 0 : Ax \geq b\}$ $(\neq \emptyset)$ is of the form $uAx + vx \geq ub - t$, where $u, v, t \geq 0$.
- Subtract a nonnegative surplus variable $\alpha x - s = \beta$.
- Generate a Gomory inequality.
- Eliminate $s = \alpha x - \beta$ to get the inequality in the x-space.
- The convex set obtained by intersecting all these inequalities with P is called the Gomory closure.

THEOREM Cook, Kannan, Schrijver 1990
The Gomory closure is a polyhedron.

THEOREM Caprara, Letchford 2002 et Cornuéjols, Li 2002
It is NP-hard to optimize a linear function over the Gomory closure.

Nevertheless,
Balas and Saxena 2006 and Dash, Günlück and Lodi 2007
were able to optimize over the Gomory closure by solving a sequence of parametric MILPs.

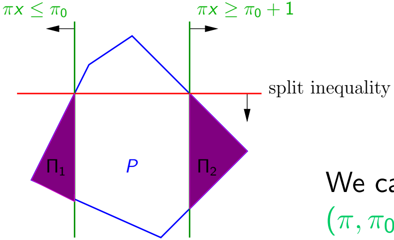
DUALITY GAP CLOSED BY DIFFERENT CUTS
MIPLIB 3

Gomory cuts (optimal basis)	Gomory closure
24 %	80 %

Split Inequalities   Cook-Kannan-Schrijver 1990

$P := \{x \in \mathbb{R}^n : Ax \geq b\}$
$S := P \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$.

For $\pi \in \mathbb{Z}^n$ such that $\pi_{p+1} = \ldots = \pi_n = 0$ and $\pi_0 \in \mathbb{Z}$, define



$$\Pi_1 := P \cap \{x : \pi x \leq \pi_0\}$$

$$\Pi_2 := P \cap \{x : \pi x \geq \pi_0 + 1\}$$

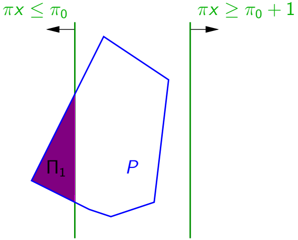
We call $cx \leq c_0$ a split inequality if there exists $(\pi, \pi_0) \in \mathbb{Z}^p \times \mathbb{Z}$ such that $cx \leq c_0$ is valid for $\Pi_1 \cup \Pi_2$.

The split closure is the intersection of all split inequalities.

THEOREM Nemhauser-Wolsey 1990, Cornuéjols-Li 2002
The split closure is identical to the Gomory closure.

Chvátal Inequalities Chvátal 1973

A Chvátal inequality is a split inequality where $\Pi_2 = \emptyset$.



The Chvátal closure is the intersection of all these inequalities.

REMARK Chvátal defined this concept in 1973 in the context of pure integer programs.

The Chvátal closure reduces the duality gap by around 63 % on the pure integer MIPLIB 03 instances (Fischetti-Lodi 2006) and around 28 % on the mixed instances (Bonami-Cornuéjols-Dash-Fischetti-Lodi 2007).

Lift-and-Project

Sherali-Adams 1990
Lovász-Schrijver 1991
Balas-Ceria-Cornuéjols 1993

Let $S := \{x \in \{0,1\}^p \times \mathbb{R}_+^{n-p} : \; Ax \geq b\}$
$P := \{x \in \mathbb{R}_+^n : \; Ax \geq b\}$

LIFT-AND-PROJECT PROCEDURE

STEP 0 Choose an index $j \in \{1, \ldots, p\}$.

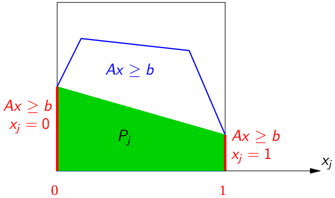
STEP 1 Generate the nonlinear system $$\begin{aligned} x_j(Ax-b) &\geq 0 \\ (1-x_j)(Ax-b) &\geq 0 \end{aligned}$$

STEP 2 Linearize the system by substituting $x_i x_j$ by y_i for $i \neq j$, and x_j^2 by x_j. Denote this polyhedron by M_j.

STEP 3 Project M_j on the x-space. Denote this polyhedron by P_j.

PROPOSITION $\text{Conv}(S) \subseteq P_j \subseteq P$.

THEOREM $P_j = \text{Conv}\{\begin{pmatrix} Ax \geq b \\ x_j = 0 \end{pmatrix} \cup \begin{pmatrix} Ax \geq b \\ x_j = 1 \end{pmatrix}\}$.



THEOREM Balas 1979 $\text{Conv}(S) = P_p(\ldots P_2(P_1)\ldots)$.

LIFT-AND-PROJECT CUT

Given a fractional solution $\bar{x}$ of the linear relaxation $Ax \geq b$, find a cutting plane $\alpha x \geq \beta$ (namely $\alpha\bar{x} < \beta$) that is valid for P_j (and therefore for S).

DEEPEST CUT
$$\max \beta - \alpha\bar{x}$$
$$\alpha x \geq \beta \text{ valid for } P_j$$

CUT GENERATION LINEAR PROGRAM

$$M_j := \{x \in \mathbb{R}^n_+, y \in \mathbb{R}^n_+ : \\ Ay - bx_j \geq 0, \\ Ax + bx_j - Ay \geq b, \\ y_j = x_j\}$$

The first two constraints come from the linearization in STEP 1.

In fact, one does not use the variable y_j

$$M_j := \{x \in \mathbb{R}^n_+, y \in \mathbb{R}^{n-1}_+ : \\ B_j x + A_j y \geq 0, \\ D_j x - A_j y \geq b\}$$

To project onto the x-space, we use the cone $Q := \{u, v \geq 0 : \ uA_j - vA_j = 0\}$

$$P_j = \{x \in \mathbb{R}^n_+ : \ (uB_j + vD_j)x \geq vb \text{ for all } (u, v) \in Q\}$$

DEEPEST CUT

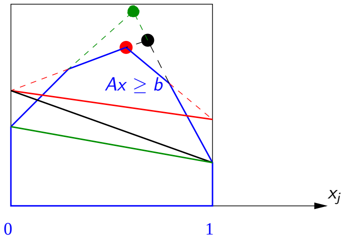
$$\max \ vb - (uB_j + vD_j)\bar{x} \\ uA_j - vA_j = 0 \\ u \geq 0, v \geq 0 \\ \sum u_i + \sum v_i = 1$$

SIZE OF THE CUT GENERATION LP

$$\begin{aligned} \max\ vb - (uB_j + vD_j)\bar{x} \\ uA_j - vA_j = 0 \\ \sum u_i + \sum v_i = 1 \\ u \geq 0, v \geq 0 \end{aligned}$$

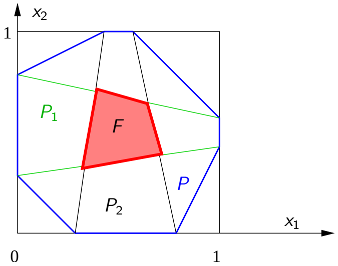
Number of variables : $2m$
Number of constraints : n + nonnegativity

Balas and Perregaard 2003 give a precise correspondance between the basic feasible solutions of the cut generation LP and the basic solutions of the LP $\begin{aligned} \min\ cx \\ Ax \geq b \end{aligned}$



LIFT-AND-PROJECT CLOSURE OF P

$$F := \bigcap_{j=1}^{p} P_j$$



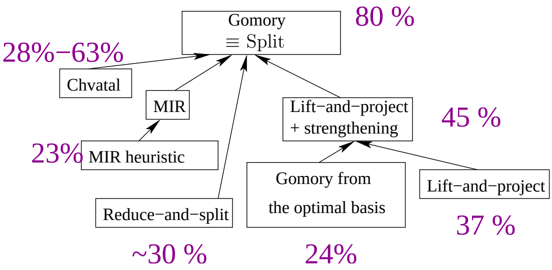
REMARK Balas and Jeroslow 1980 show how to strengthen cutting planes by using the integrality of the other integer variables (lift-and-project only considers the integrality of one x_j at a time).

Experiments of Bonami and Minoux 2005 on MIPLIB03 instances :

Gap closed

Lift-and-project closure	Lift-and-project + strengthening
37 %	45 %

Duality gap closed by different types of cutting planes
MIPLIB 3 instances



All these cuts are generated from integrality arguments applied to one linear equation. Can we generate deeper cuts by considering several equations ?

Corner Polyhedron [Gomory 1969]

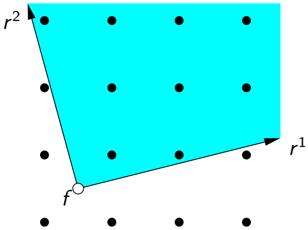
Relax nonnegativity on basic variables x_j.

In our current work [Basu, Bonami, Borozan, Conforti, Cornuéjols, Margot, Zambelli 2009], we make a further relaxation :

Relax integrality on nonbasic variables.

$$\begin{aligned} x &= f + \textstyle\sum_{j=1}^{k} r^j s_j \\ x &\in \mathbb{Z}^q \\ s &\geq 0 \end{aligned}$$

Example



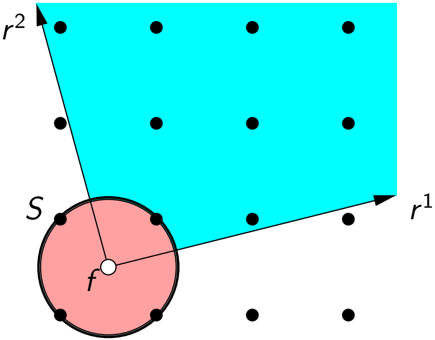
Feasible set $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{Z}^2 : \right.$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = f + r^1 s_1 + r^2 s_2$$

where $\left. s_1 \geq 0, s_2 \geq 0 \right\}$

Intersection Cuts [Balas 1971]

Assume $f \notin \mathbb{Z}^q$. Want to cut off the basic solution $s = 0, x = f$.



Any convex set S with $f \in int(S)$ with no integer point in $int(S)$.

Intersection Cuts [Balas 1971]

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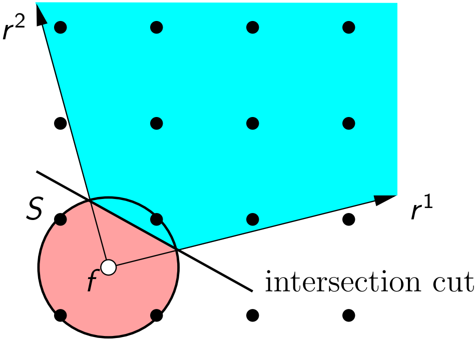
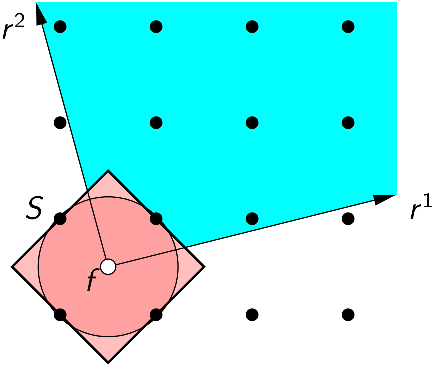


Any convex set S with $f \in int(S)$ with no integer point in $int(S)$.
Compute intersection of the rays with the boundary of S.
Cut defined by these points is valid : $\psi(r^1)s_1 + \psi(r^2)s_2 \geq 1$.

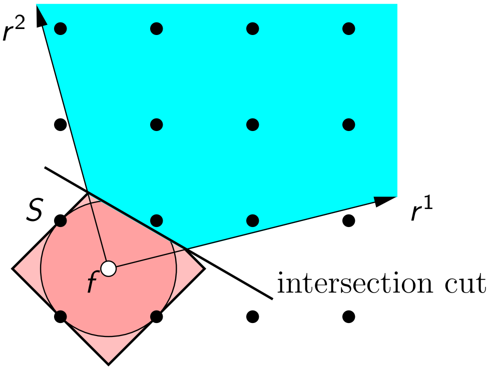
A Better Intersection Cut [Balas 1971]



Bigger convex set :

Octahedron $f \in int(S)$ with no integral point in $int(S)$.

A Better Intersection Cut [Balas 1971]



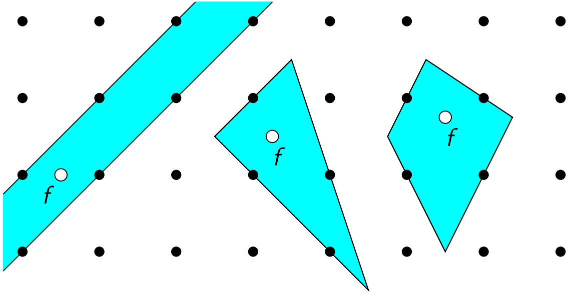
Bigger convex set :

Octahedron $f \in int(S)$ with no integral point in $int(S)$.

Better cut : $\psi(r^1)s_1 + \psi(r^2)s_2 \geq 1$.

Maximal Lattice-Free Convex Sets in the Plane

Split, triangles and quadrilaterals



generate split, triangle and quadrilateral inequalities $\sum \psi(r) s_r \geq 1$.

Split closure := the intersection of all split inequalities.
Triangle closure := the intersection of all inequalities arising from maximal lattice-free triangles.
Quadrilateral closure := the intersection of all inequalities arising from maximal lattice-free quadrilaterals.

Since all the facets of Integer Hull are induced by these three families of maximal lattice-free convex sets (Andersen, Louveaux, Wolsey, Weismantel 2007), we have

Integer Hull = Split closure $\cap$ Triangle closure $\cap$ Quad closure

The split closure is a polyhedron (Cook, Kannan and Schrijver) but such a result is not known for the triangle closure and the quadrilateral closure.

THEOREM Basu, Bonami, Cornuéjols, Margot 2008
Triangle closure $\subseteq$ Split closure and
Quad closure $\subseteq$ Split closure.

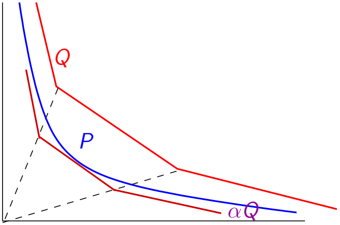
A theorem of Goemans 1995

Let $Q := \{x : a^i x \geq b_i \text{ for } i = 1, \ldots, m\} \subseteq \mathbb{R}^n_+ \setminus \{0\}$ where $a^i \geq 0$ and $b_i \geq 0$ for all i.
For $\alpha > 0$ let $\alpha Q := \{x : \alpha a^i x \geq b_i \text{ for } i = 1, \ldots, m\}$.

THEOREM If convex set $P \subseteq \mathbb{R}^n_+$ contains Q, then the smallest value of $\alpha \geq 1$ such that $P \subseteq \alpha Q$ is

$$\max_{i=1,\ldots,m} \left\{ \frac{b_i}{\inf\{a^i x \ : \ x \in P\}} \ : \ b_i > 0 \right\}.$$

In other words, the only directions that need to be considered to compute α are those defined by the nontrivial facets of Q.



Relative strength of closures
Basu, Bonami, Cornuéjols, Margot 2008

Both the triangle closure and the quadrilateral closure are good approximations of the integer hull :

THEOREM
Integer Hull $\subseteq$ Triangle closure $\subseteq$ 2 (Integer Hull) and
Integer Hull $\subseteq$ Quad closure $\subseteq$ 2 (Integer Hull)

We also show that the split closure may not be a good approximation of the integer hull :

THEOREM
For any $\alpha > 1$, there is a choice of f, $r^1, \ldots, r^k$ such that
Split closure $\not\subseteq$ α (Integer hull).

Papers available on http ://integer.tepper.cmu.edu/