Vehicle Routing with Heterogeneous Fleets and Uncertain Demands

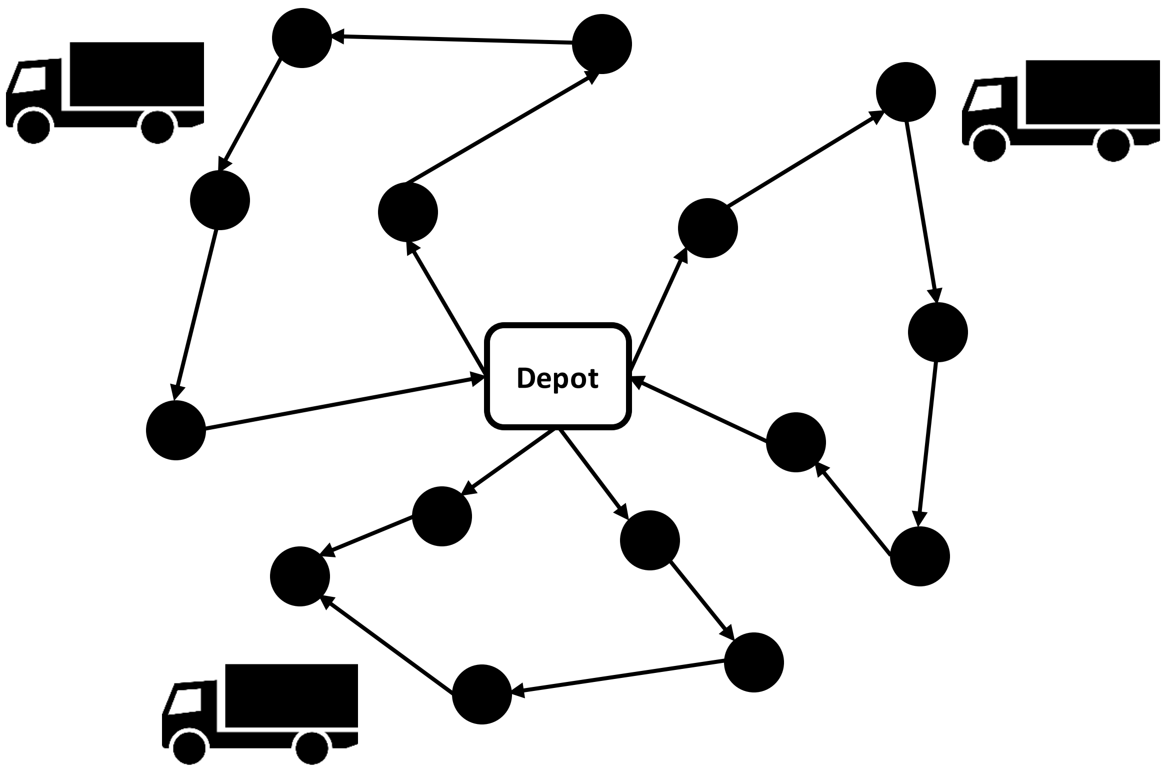
Anirudh Subramanyam, Chrysanthos E. Gounaris
Carnegie Mellon University

Laercio Oliveira, John Wassick
The Dow Chemical Company

EWO Meeting
September 30th – October 1st, 2015

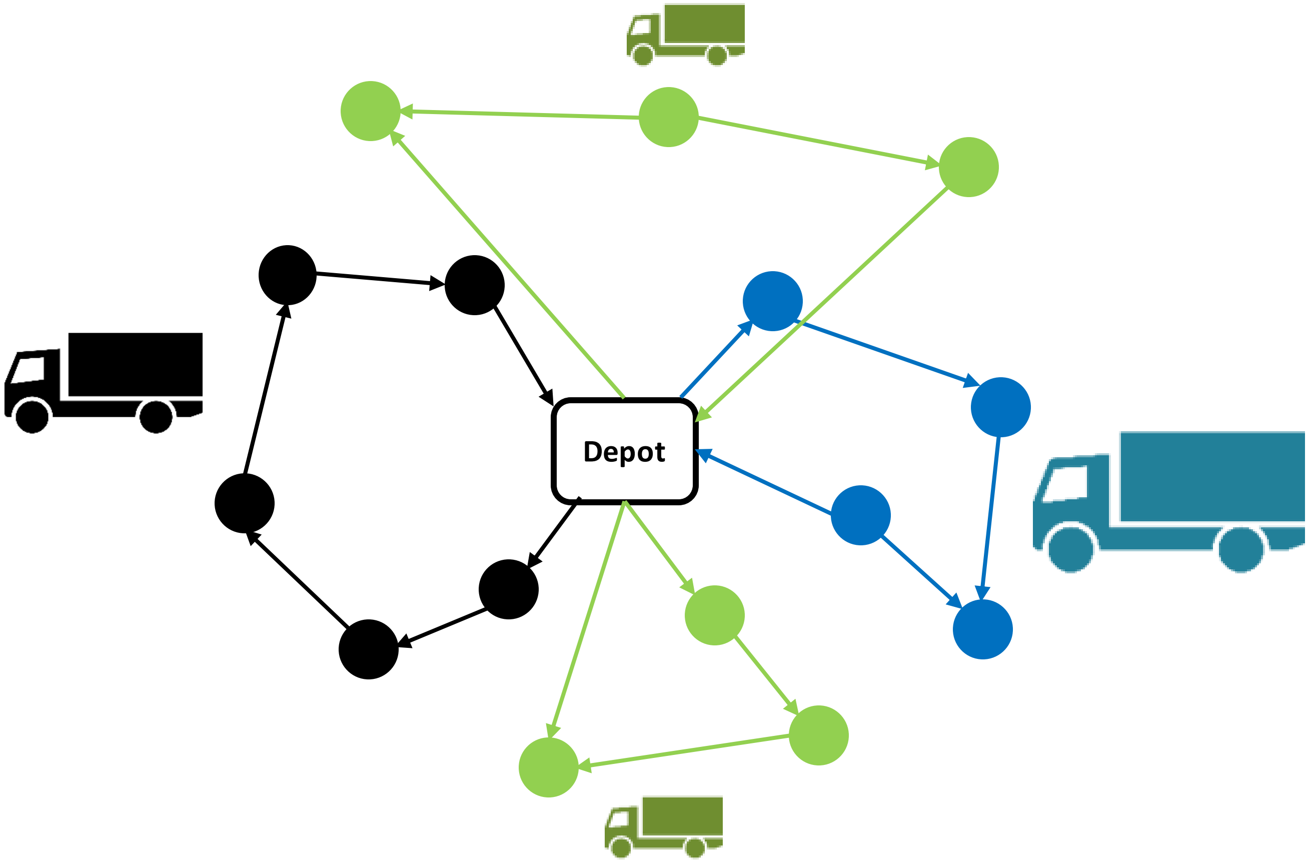
Vehicle Routing Problem (VRP)

- Given a set of customers, determine minimum cost vehicle routes such that all customer orders are satisfied
  - Fleet typically assumed to be composed of identical vehicles



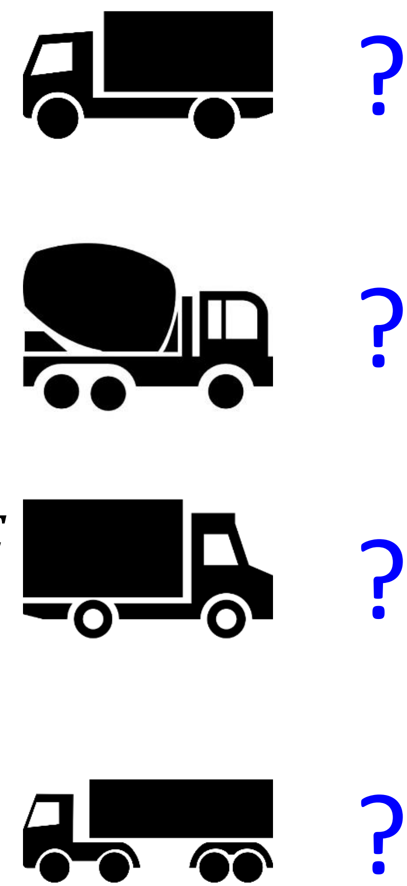
Heterogeneous Fixed Fleet VRP

- In practice, real-world fleets are very often composed of heterogeneous vehicles with different capacities and routing costs



Fleet Sizing and Mix VRP

- Deciding an optimal fleet composition and size cannot be made agnostically of vehicle routing



- Given
  - Customer set V_C with demands q_i
  - Vehicle types $k = 1, \dots, m$
  - Capacity Q_k, availability m_k, fixed cost F_k of type k (F_k could model rental/capital amortization costs)
  - Routing cost c_{ij}^k between every pair of nodes $(i, j) \in E$
- Determine a set of routes for each vehicle such that
  - Each customer is visited exactly once
  - No more than m_k vehicles of type k are used
  - Sum of routing and fixed costs is minimized

Broad Class of Heterogeneous VRPs

- Other real-world distribution problems can also be modeled as Heterogeneous VRPs (Baldacci et. al., *Ann. Oper. Res.,* 2010)

VRP Variant	Fleet Size	Fixed Costs	Routing Costs
Heterogeneous Fleet	Limited	Not considered	Vehicle-dependent
Site Dependent	Limited	Not considered	Site-dependent
Fleet Size and Mix	Unlimited	Considered	Independent
Multi-depot	Unlimited	Not considered	Depot-dependent

- None of the existing approaches account for uncertainty in available information

- Current state-of-the-art is based on a *set partitioning* approach
  - Not straightforward to incorporate uncertainty in this approach

New "Undirected" Formulation I

- $y_i^k \in \{0,1\}$ denotes if customer i is assigned to a vehicle of type k
- $x_{ij}^k \in \{0,1\}$ denotes if edge (i,j) is traveled by a vehicle of type k
  - $\mathcal{O}(n^2 M)$ binary variables and $\mathcal{O}(2^n)$ constraints

$$\underset{x,y}{\text{minimize}} \quad \sum_k \left(\sum_{(i,j)\in E} c_{ij} x_{ij}^k + F_k \sum_{i \in V_C} x_{0i}^k \right)$$

subject to

Degrees

$$\sum_{\substack{j\in V:\\ j>i}} x_{ij}^k + \sum_{\substack{j\in V:\\ j<i}} x_{ji}^k = 2y_i^k \qquad \forall\, i \in V_C,\ \forall\, k$$

$$\sum_{j\in V_C} x_{0j}^k = \sum_{j \in V_C} x_{j,n+1}^k \leq m_k \qquad \forall\, k$$

Assignment

$$\sum_k y_i^k = 1 \qquad \forall\, i \in V_C$$

Subtour elimination

$$\sum_{(i,j)\in\delta(S)} x_{ij}^k \geq 2y_v^k \qquad \forall\, v \in S,\ \forall\, S \subseteq V_C,\ \forall\, k$$

"Generalized" RCI

$$\sum_{(i,j)\in\delta(S)} x_{ij}^k + 2\sum_{i\in S}(1-y_i^k) \geq 2\left\lceil \frac{1}{Q_k}\sum_{i\in S} q_i \right\rceil \qquad \forall\, S \subseteq V_C,\ \forall\, k$$

New “Undirected” Formulation II

- Define extended graph (V', E'), where $V' = V \cup \{n+1, \dots, n+m\}$
  Each customer is connected to m additional destination depots
  - Cost of edges $(i, n+k)$ is $c'_{i,n+k} = c_{i0} + F_k$
  - Other edges have same cost $c'_{ij} = c_{ij}$

- $y_i^k \in [0,1]$ denotes if customer i is assigned to a vehicle of type k
- $x_{ij} \in \{0,1\}$ denotes if edge (i,j) is used
  - $\mathcal{O}(n^2 + nM)$ binary variables, $\mathcal{O}(nM)$ continuous variables and $\mathcal{O}(2^n)$ constraints

- This model uses aggregated variables and, hence,…
  …is of smaller size than Formulation I
  …has weaker LP relaxation than Formulation I

New "Undirected" Formulation II

$$\underset{x,y}{\text{minimize}} \quad \sum_{(i,j)\in E'} c'_{ij} x_{ij}$$

subject to

Degrees

$$\sum_{j\in V_C} x_{0j} = \sum_k \sum_{j\in V_C} x_{j,n+k}$$

$$\sum_{\substack{j\in V':\\ j>i}} x_{ij} + \sum_{\substack{j\in V':\\ j<i}} x_{ji} = 2 \qquad \forall\, i \in V_C$$

Fleet Availability →

$$\sum_{j\in V_C} x_{j,n+k} \le m_k \qquad \forall\, k$$

Subtour elimination

$$\sum_{(i,j)\in\delta(S)} x_{ij} \ge 2\left\lceil \frac{1}{\max_k Q_k} \sum_{i\in S} q_i \right\rceil \qquad \forall\, S \subseteq V_C$$

$$\sum_{(i,j)\in\delta(S)} x_{ij} \ge \sum_{j\in V_C} x_{0j} \qquad \forall\, S \subseteq V_C \cup \{0\} : 0 \in S$$

"Generalized" RCI →

$$\sum_{(i,j)\in\delta(S)} x_{ij} + 2\sum_{i\in S}\left(1 - y_i^k\right) \ge 2\left\lceil \frac{1}{Q_k} \sum_{i\in S} q_i \right\rceil \qquad \forall\, S \subseteq V_C,\ \forall\, k$$

Assignment

$$\sum_k y_i^k = 1 \qquad \forall\, i \in V_C$$

$$y_i^k \ge x_{i,n+k} \qquad \forall\, i \in V_C,\ \forall\, k$$

$$1 - x_{ij} \ge \max\left\{y_i^k - y_j^k, y_j^k - y_i^k\right\} \qquad \forall\, i,j \in V_C : i < j,\ \forall\, k$$

Demand Uncertainty

- In practice, customer demands are often not known with certainty
  - Deterministic routing plan can become infeasible or too expensive
  - Importance is amplified for Fleet Sizing and Mix problems:
    not accounting for uncertainty can lead to high rental/capital costs

- Objective is to design minimum cost routing plan that remains feasible for all demand realizations within the "uncertainty set"

$$\mathcal{Q} = \{\boldsymbol{q} \in \mathbb{R}^n_+ : W\boldsymbol{q} \leq h\}$$

- Practically-meaningful uncertainty sets:

Inclusion-constrained budgets

$$\mathcal{Q}_B = \left\{\boldsymbol{q} \in [\underline{\boldsymbol{q}}, \bar{\boldsymbol{q}}] : \sum_{i \in B_l} q_i \leq b_l, \quad \forall\, l = 1, \ldots, L\right\}$$

$$B_l \subseteq B_{l'} \text{ or } B_{l'} \subseteq B_l \text{ or } B_l \cap B_{l'} = \emptyset \quad \forall\, l, l'$$

Beta-net-alpha factor model

$$\mathcal{Q}_F = \{\boldsymbol{q} \in [\underline{\boldsymbol{q}}, \bar{\boldsymbol{q}}] : \exists \xi \in \Xi : \boldsymbol{q} = \boldsymbol{q}^{\mathbf{0}} + \Phi\xi\}$$

$$\text{where } \Xi = \left\{\xi \in [-1, 1]^F : |\sum_{f=1}^{F} \xi_f| \leq \beta F\right\}$$

Robust Counterpart

- All constraints remain exactly the same except "Generalized" RCI

$$\textbf{(F1)} \sum_{(i,j)\in\delta(S)} x_{ij}^k + 2\sum_{i\in S}(1-y_i^k) \geq 2\left\lceil \frac{1}{Q_k}\sum_{i\in S} q_i \right\rceil \quad \forall\, S \subseteq V_C,\ \forall\, k, \forall\, \boldsymbol{q} \in \mathcal{Q}$$

- Reformulated to obtain (similar for Formulation 2)
  Robust Generalized RCI cuts:

$$\textbf{(F1)} \sum_{(i,j)\in\delta(S)} x_{ij}^k + 2\sum_{i\in S}(1-y_i^k) \geq 2\left\lceil \frac{1}{Q_k}\max_{\boldsymbol{q}\in\mathcal{Q}}\sum_{i\in S} q_i \right\rceil \quad \forall\, S \subseteq V_C,\ \forall\, k$$

- Can obtain closed-form solutions of $\max_{\boldsymbol{q}\in\mathcal{Q}}\sum_{i\in S} q_i$ for inclusion-constrained budgets and beta-net-alpha factor models (Gounaris et. al., *Oper. Res.,* 2013)
  - Can be used efficiently in a separation routine for Robust Generalized RCI cuts

Separation of Valid Robust Inequalities

- Generalized RCI cuts separated using Tabu Search *metaheuristic*
  - Construct an initial set $S \subseteq V_C$ using a *greedy heuristic*
  - Update S *incrementally* to maximize violation
  - Incremental steps require repeated calculation of $\max_{\boldsymbol{q}\in\mathcal{Q}} \sum_{i\in S} q_i$ which is immediate because of closed-form solutions

- Generalized FCI cuts (not necessary, but strengthen linear relaxation)

$$\textbf{(F1)} \qquad \sum_{(i,j)\in\delta(S)} x_{ij}^k \geq 2 \sum_{i\in S} \frac{q_i}{Q_k} y_i^k \quad \forall\, S \subseteq V_C,\ \forall\, k, \forall\, \boldsymbol{q} \in \mathcal{Q}$$

$$\textbf{(F2)} \qquad \sum_{(i,j)\in\delta(S)} x_{ij} \geq 2 \sum_{k} \sum_{i\in S} \frac{q_i}{Q_k} y_i^k \quad \forall\, S \subseteq V_C,\ \forall\, k, \forall\, \boldsymbol{q} \in \mathcal{Q}$$

  - For a given extreme point $\hat{q}$ of $\mathcal{Q}$, find maximally violated FCI $\hat{S} \subseteq V_C$ by solving an appropriate max-flow problem (polynomial time)
  - Solve an LP over $\mathcal{Q}$ with fixed $\hat{S}$ to get $q \in \mathcal{Q}$ whose FCI is most violated
  - Store in memory the generated extreme points for future separation

Conclusions and Future Work

- We developed two new deterministic MILP models for a broad class of industrially-relevant Heterogeneous VRPs
  - Generalized the classical RCI and FCI cuts
- We have developed Robust Counterparts of these models which have the same size as the original models.
- We developed efficient separation procedures for the Robust Generalized RCI cuts and Robust Generalized FCI cuts
- Future work:
  - Improve the strength of the aggregated formulation
  - Better RCI-like cuts for the deterministic and robust models
  - Explore cross-vehicle decomposition techniques
  - Apply models to a Dow case study