Tutorial on Lagrangean Decomposition: Theory and Applications

Ignacio Grossmann and Bora Tarhan
Center for Advanced Process Decision-making
Department of Chemical Engineering
Carnegie Mellon University
Pittsburgh, PA 15213

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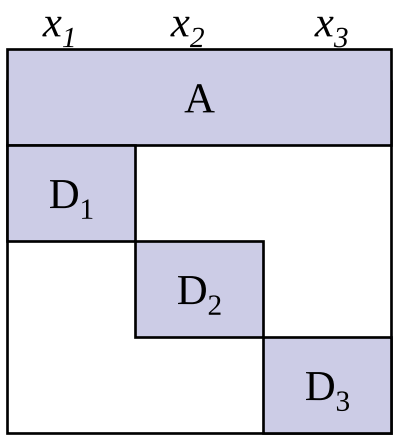
Decomposable MILP Problems

Complicating Constraints

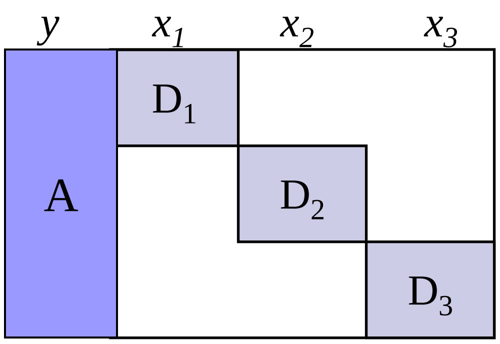


complicating constraints

$$\max \ c^T x$$
$$st \ \ Ax = b$$
$$D_i x_i = d_i \ \ i = 1,..n$$
$$x \in X \ = \{x \mid x_i \, , i = 1,..n, \mid x_i \geq 0\}$$

Lagrangean decomposition

Complicating Variables



complicating constraints

$$\max \ a^T y + \sum_{i=1,..n} c_i^T x_i$$
$$st \ \ Ay + D_i x_i = d_i \ \ i = 1,..n$$
$$y \geq 0, \ x_i \geq 0, i = 1,..n$$

Benders decomposition

Note: can reformulate by defining

$y_i = y_{i+1}$ **Complicating constraints**

and apply Lagrangean decomposition

About Lagrange

Joseph Louis Lagrange
January 25, 1736 – April 10, 1813

Born in Turin, Italy: Italian parents (French ancestors father side)

Born: Giuseppe Lodovico Lagrangia

1766: Lagrange succeeded Euler as director of mathematics at the Prussian Academy of Sciences in Berlin
1794: Became the first professor of analysis at the opening of École Polytechnique
1808: Napoleon named him to the Legion of Honour and made him a Count of the Empire in 1808

Lagrangean or Lagrangian Decomposition?

Google hits:
– "Lagrangian" returned 2,190,000 hits.
– "Lagrangean" returned 104,000 hits.

Google hits:
– "Lagrangian decomposition" returned 671,00 hits.
– "Lagrangean decomposition" returned 1,580,000 hits.

Google hits:
– "Lagrangian relaxation" returned 133,000 hits.
– "Lagrangean relaxation" returned 275,000 hits.

We will spell it as <u>"Langrangean"</u>

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Lagrangean Relaxation *(Fisher, 1985)*

- **MILP optimization problems can often be modeled as problems with complicating constraints.**

- **The complicating constraints are added to the objective function (i.e. dualized) with a penalty term (Lagrangean multiplier) proportional to the amount of violation of the dualized constraints.**

- **The Lagrangean problem is easier to solve *(eg. can be decomposed)* than the original problem and provides an upper bound to a maximization problem.**

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Lagrangean Relaxation

$$Z = \max \quad cx$$

$$Ax \le b$$

$$Dx \le e \qquad \textbf{(IP)}$$

$$x \in Z_+^n$$

Assume integers only
Easily extended cont. vars.

Assume that $Ax \le b$ **is complicating constraint**

$$Z_{LR}(u) = \max \quad cx + u(b - Ax)$$

$$Dx \le e$$

$$x \in Z_+^n$$

$$where \qquad u \ge 0 \quad Lagrange\ multipliers$$

Lagrangean Relaxation

$$Z = \max \quad cx \qquad\qquad Z_{LR}(u) = \max \quad cx + u(b - Ax)$$

Complicating Constraint $\longrightarrow$ $Ax \le b$

$$Dx \le e \qquad\qquad Dx \le e$$

$$x \in Z_+^n \qquad\qquad x \in Z_+^n$$

$$\textit{where} \quad u \ge 0$$

This is a relaxation of original problem because:

i) removing the constraint $Ax \le b$ **relaxes the original feasible space,**

ii) $Z_{LR}(u) \ge Z$ **always holds as in the original space since** $(b - Ax) \ge 0$

and Lagrange multiplier is always $u \ge 0$**.**

Lagrangean Relaxation Yields Upper Bound $\Rightarrow$ $Z_{LR}(u) \ge Z$

Lagrangean Relaxation

Original problem:

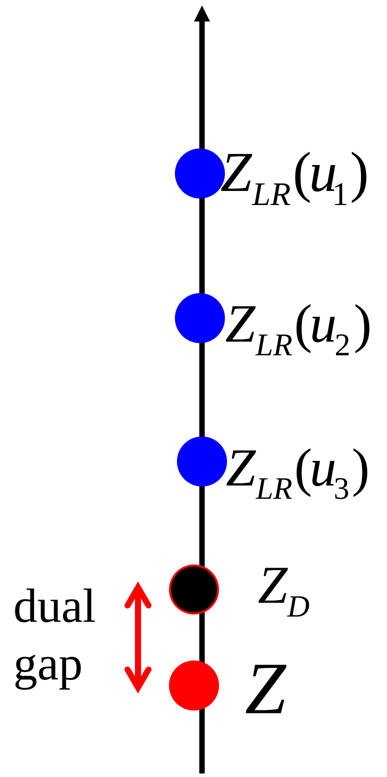
$$
\begin{aligned}
Z = \max \quad & cx \\
& Ax \le b \\
& Dx \le e \\
& x \in Z_+^n
\end{aligned}
$$

Relaxed problem:

$$
\begin{aligned}
Z_{LR}(u) = \max \quad & cx + u(b - Ax) \\
& Dx \le e \\
& x \in Z_+^n
\end{aligned}
$$

Lagrangean dual:

$$
\begin{aligned}
Z_D = \min\, & Z_{LR}(u) \\
& u \ge 0
\end{aligned}
$$



Graphical Interpretation

Relaxed problem:

$$\boxed{\begin{aligned} Z_{LR}(u) = \max \quad & cx + u(b - Ax) \\ & Dx \le e \\ & x \in Z_+^n \end{aligned}}$$

Lagrangean dual:

$$\begin{aligned} Z_D = \min\, & Z_{LR}(u) \\ & u \ge 0 \end{aligned}$$

Combine Relaxed and Lagrangean Dual Problems:

$$\begin{aligned} Z_D = \min_{u\ge 0} \Big\{ \max_{x \ge 0} \quad & cx + u(b - Ax) \Big\} \\ & Dx \le e \\ & x \in Z_+^n \end{aligned}$$

Graphical Interpretation

$$Z_D = \min_{u\geq 0}\left\{\max_{x\geq 0} \quad cx + u(b - Ax)\right\}$$
$$Dx \leq e$$
$$x \in Z_+^n$$

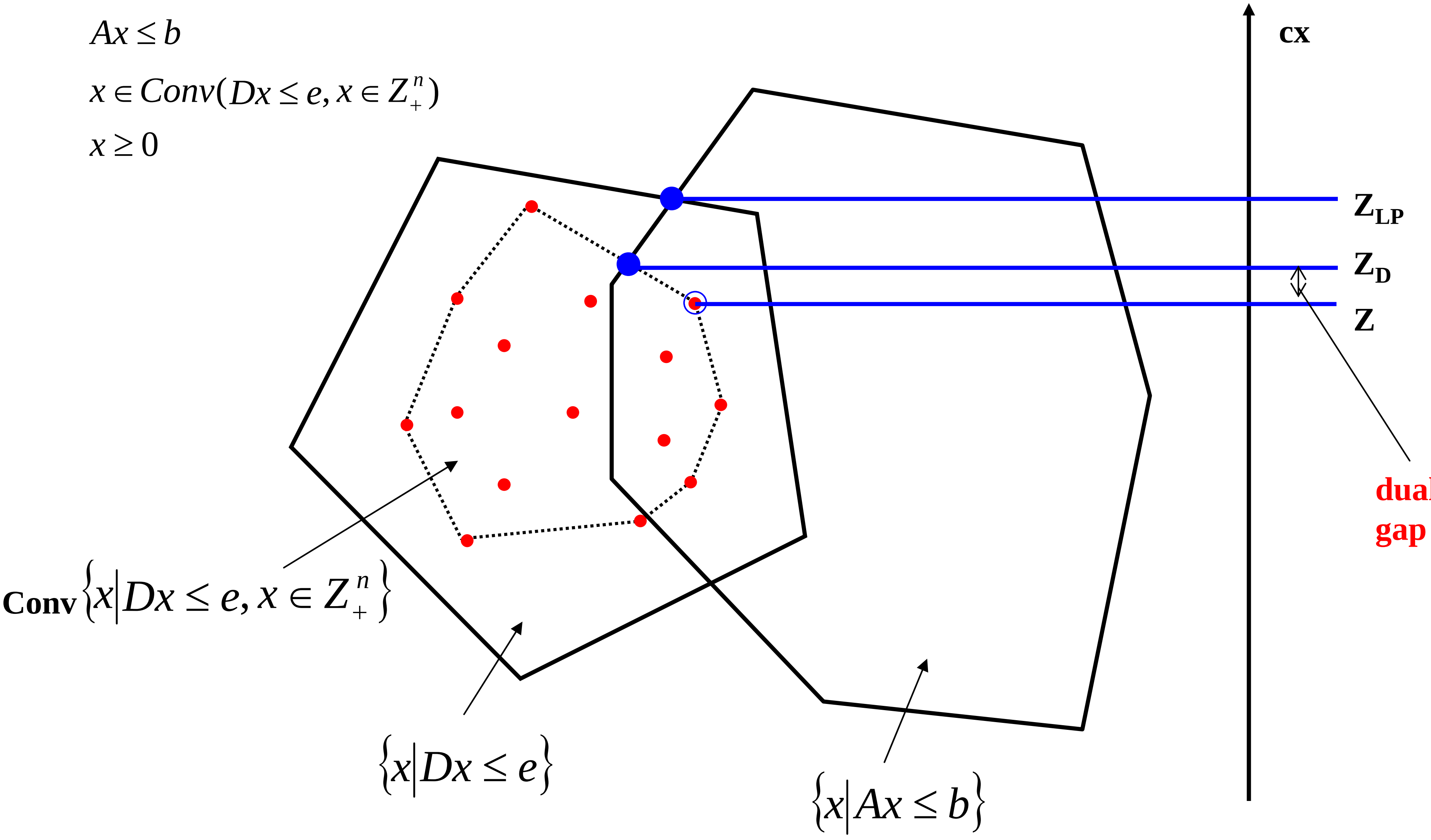
Nice Proof
Frangioni (2005)

$$Z_D{}' = \max cx$$
$$Ax \leq b$$
$$x \in Conv(Dx \leq e, x \in Z_+^n)$$
$$x \geq 0$$

Optimization of Lagrange multipliers (dual) can be interpreted as optimizing the primal objective function on the intersection of the convex hull of non-*complicating* constraints set $\{x \mid Dx \leq e, x \in Z_+^n\}$ **and the LP relaxation of the *relaxed* constraints set** $\{x \mid Ax \leq b, x \in Z_+^n\}$ **.**

Graphical Interpretation

$$Z_D' = \max cx$$
$$Ax \le b$$
$$x \in Conv(Dx \le e, x \in Z_+^n)$$
$$x \ge 0$$



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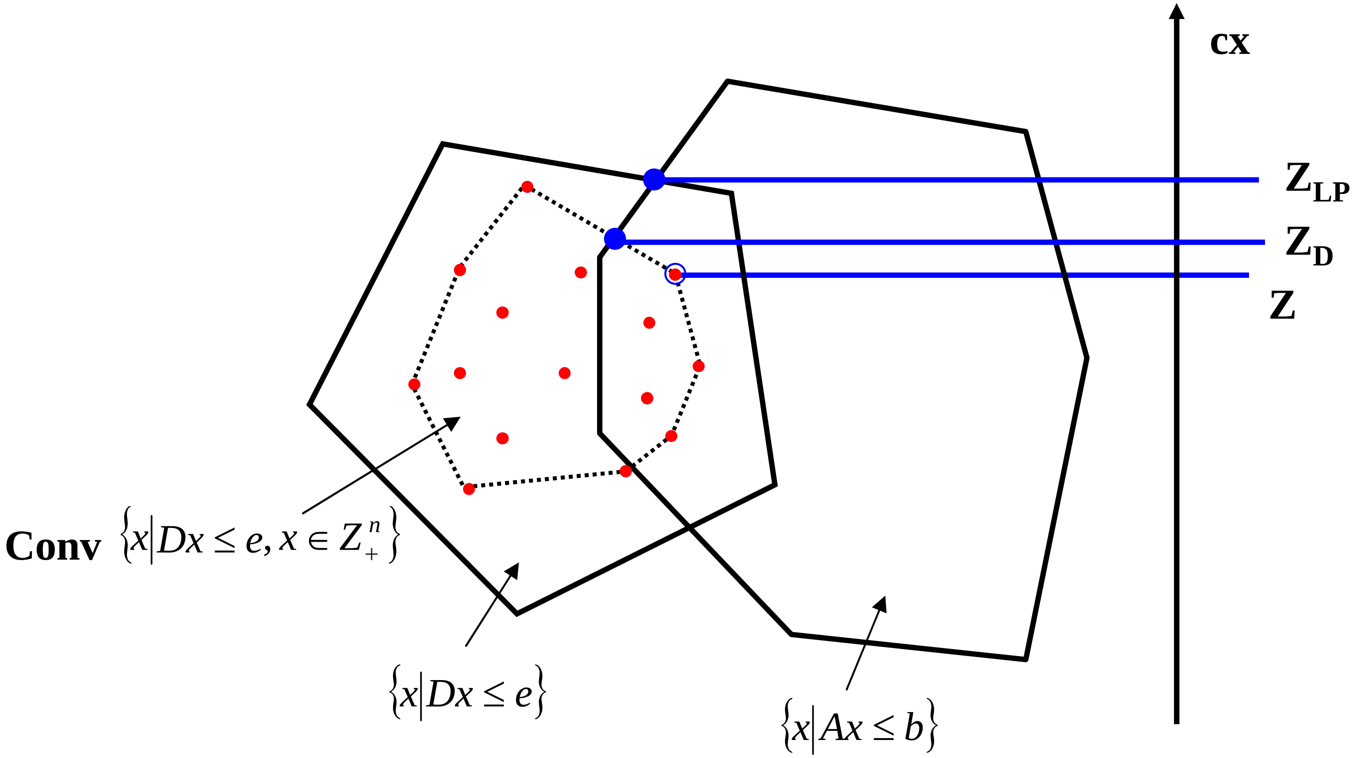
Theorem

Lagrangean relaxation yields a bound at least as tight as LP relaxation

$$Z(P) \le Z_D \le Z_{LR}(u) \le Z_{LP}$$



Lagrangean Decomposition *(Guignard & Kim, 1987)*

- **Lagrangean Decomposition is a special case of Lagrangean Relaxation.**
- **Define variables for each set of constrain, add constraints equating different variables *(new complicating constraints)* to the objective function with some penalty terms.**

$$
\begin{aligned}
Z = \max \quad & cx \\
& Ax \le b \\
& \boxed{Dx \le e} \\
& x \in Z_+^n
\end{aligned}
\qquad
\begin{aligned}
Z' = \max \quad & cx \\
& Ax \le b \\
& \boxed{Dy \le e} \\
& x = y \\
& x \in Z_+^n \\
& y \in Z_+^n
\end{aligned}
\qquad
\begin{aligned}
Z_{LD}(v) = \max \quad & cx + v(y - x) \\
& Ax \le b \\
& Dy \le e \\
& x \in Z_+^n \\
& y \in Z_+^n
\end{aligned}
$$

New complicating constraints

Dualize $x = y$

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Lagrangean Decomposition

$$
\begin{aligned}
Z_{LD}(v) = \max \quad & cx + v(y - x) \\
& Ax \le b \\
& Dy \le e \\
& x \in Z_+^n \\
& y \in Z_+^n
\end{aligned}
$$

Subproblem 1

$$
\begin{aligned}
Z_{LD1}(v) = \max \quad & (c - v)x \\
& Ax \le b \\
& x \in Z_+^n
\end{aligned}
$$

Subproblem 2

$$
\begin{aligned}
Z_{LD2}(v) = \max \quad & vy \\
& Dy \le e \\
& y \in Z_+^n
\end{aligned}
$$

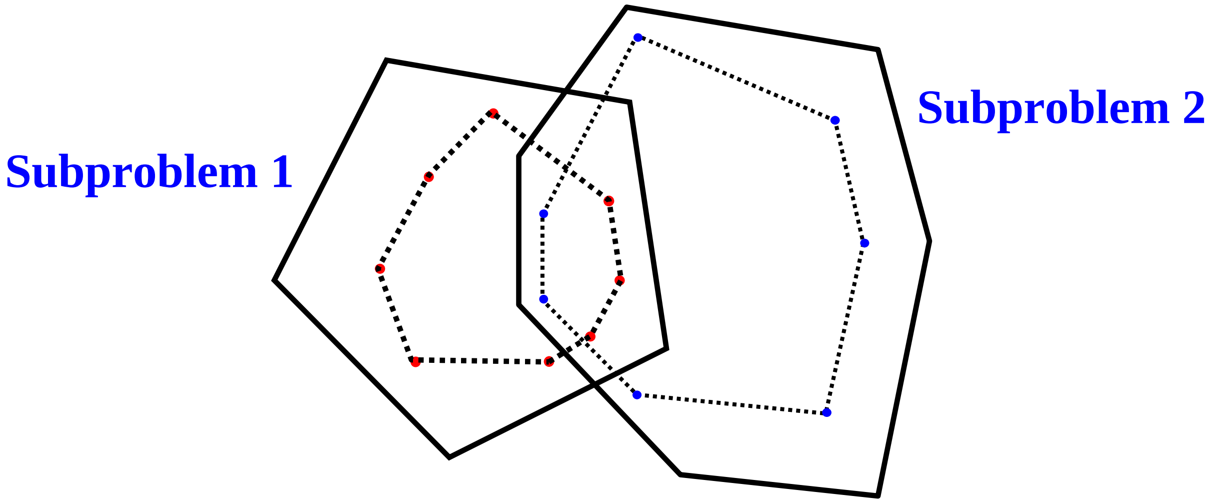
$$Z_{LD} = \min_{v \ge 0} \quad \left(Z_{LD1}(v) + Z_{LD2}(v) \right)$$

Lagrangean dual

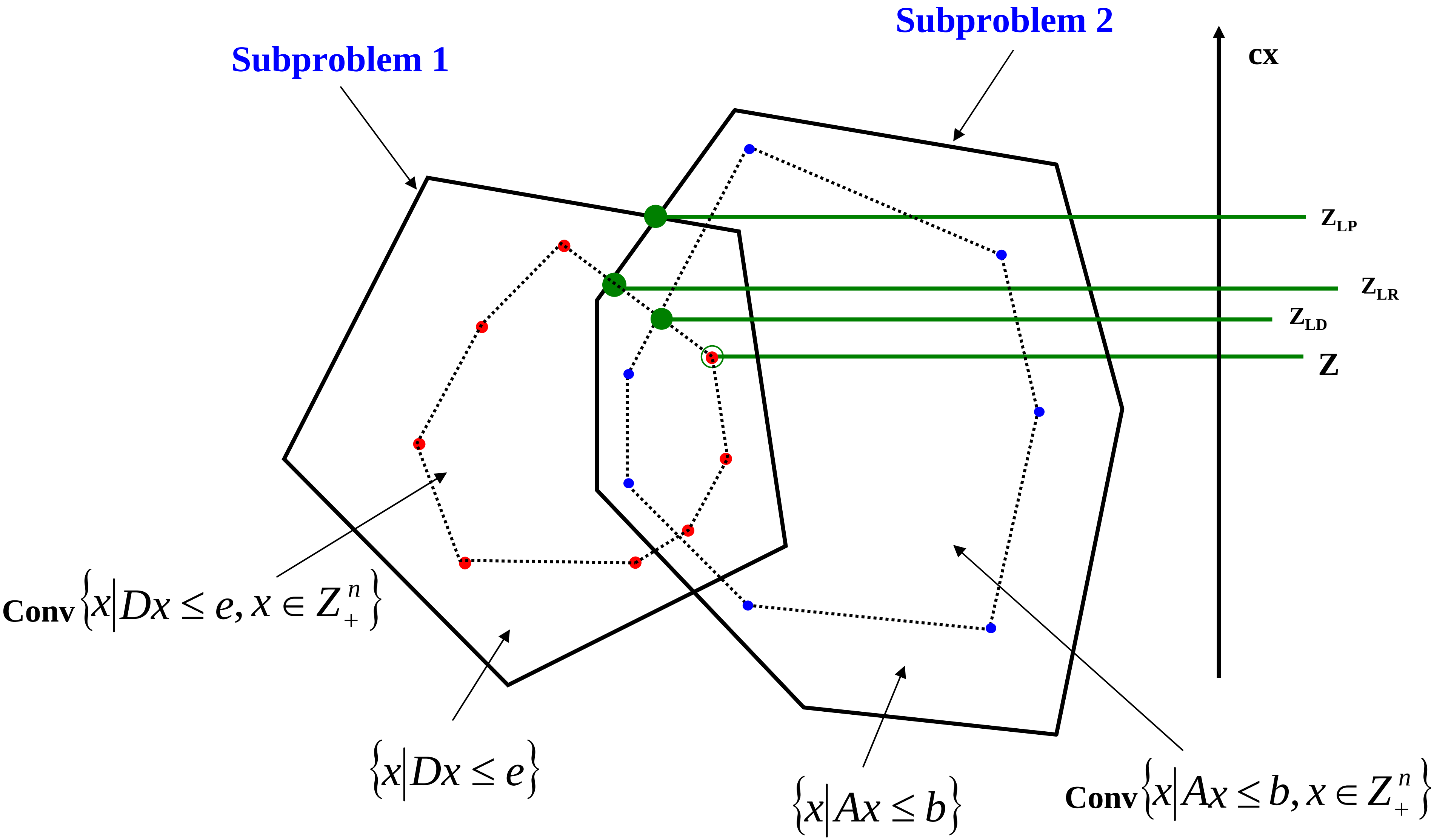
Notes

- **Lagrangean decomposition** is different from other possible relaxations because **every constraint in the original problem appears in one of the subproblems**.

<u>Graphically:</u> **The optimization of Lagrangean multipliers can be interpreted as optimizing the primal objective function on the intersection of the convex hulls of constraint sets.**



Graphical Interpretation?



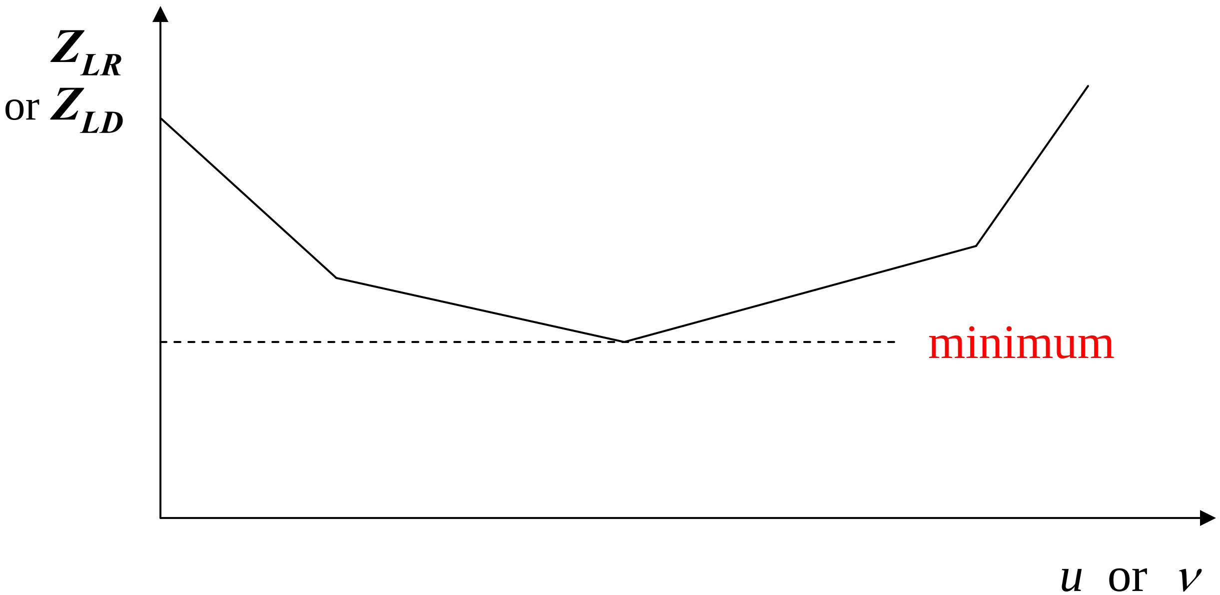
Note: Z_{LR}, Z_{LD} refer to dual solutions

Theorem

- **The bound predicted by "Lagrangean decomposition" is at least as tight as the one provided by "Lagrangean relaxation"** ***(Guignard and Kim, 1987)***
- **For a maximization problem**

$$Z(P) \leq Z_{LD} \leq Z_{LR} \leq Z_{LP}$$

Solution of Dual Problem



Piecewise linear

$\Downarrow$

Non-differentiable

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How to iterate on multipliers u?

CAPD CENTER

Assuming $Dx \le d$ is a bounded polyhedron (polytope) with extreme points

$x^k \; k = 1, 2 \ldots K$ **, then**

$$\max_x \{ cx + u(b - Ax) \mid Dx \le d, x \in X \} = \max_{k=1,\ldots K} \{ cx^k + u(b - Ax^k) \}$$

$\Downarrow$ **Dual problem**

$$\min_{u \ge 0} \max_{k=1,..K} \{ cx^k + u(b - Ax^k) \} = \min_{u \ge 0} \{ \eta \mid \eta \ge cx^k + u(b - Ax^k), k = 1,..K \}$$

Cutting plane approach

$$\min \; \eta$$

subgradient

$$s.t. \; \eta \ge cx^k + u(b - Ax^k), \; k = 1,..K_n$$

$$u \ge 0, \; \eta \in R^1$$

K_n = no. extreme points iteration n

Note: x^k generated from *max{cx + u^k(b-Ax)* subproblems

Subgradient Optimization Approach

Subgradient $s^k = (b - Ax^k)$

Steepest ascent search $u^{k+1} = u^k + \mu s^k$

Update formula for multipliers *(Fisher, 1985)*

$$u^{k+1} = u^k + \alpha_k (Z^{LB} - Z_{LD}{}^k)(b - Ax^k) / \left\| b - Ax^k \right\|^2$$

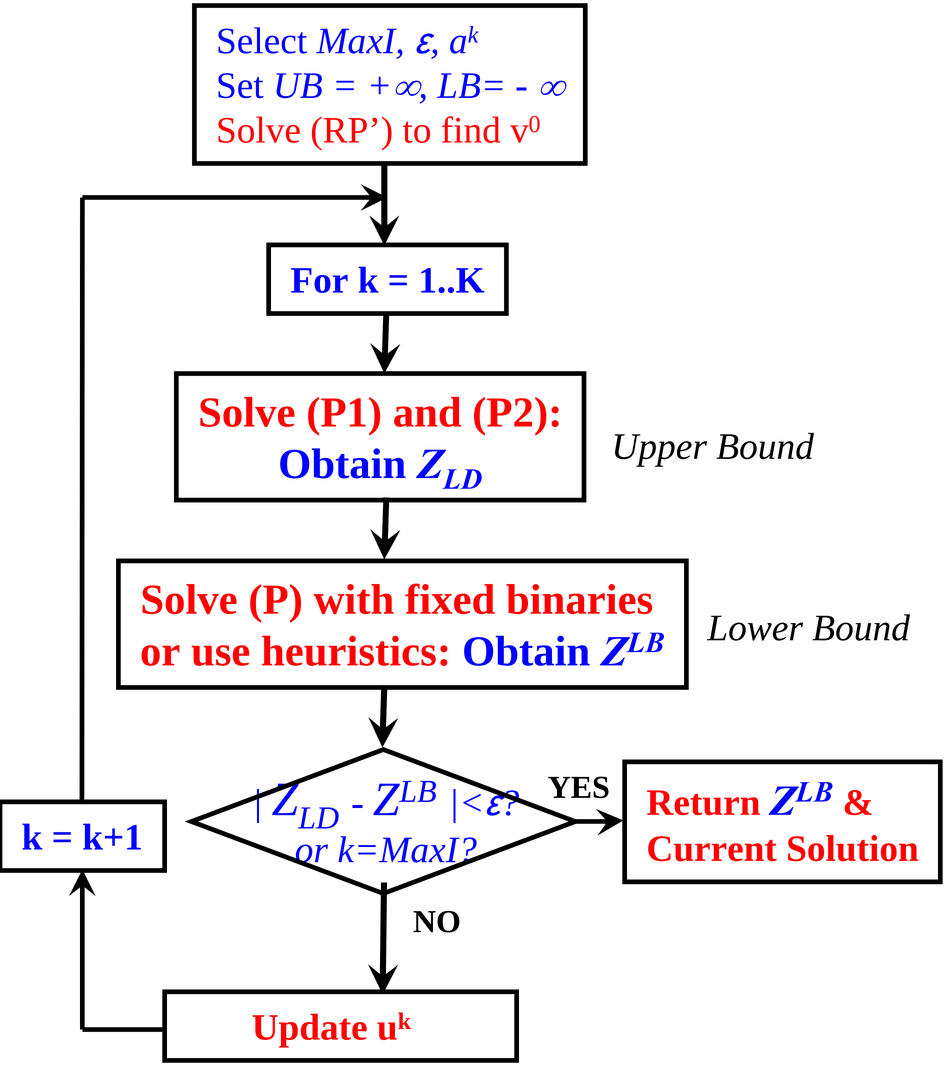
$$where\ \alpha_k \in [0, 2]$$

Note: Can also use bundle methods for nondifferentiable optimization

Lemarechal, Nemirovski, Nesterov (1995)

Solution of Langrangean Decomposition

1. Iterative search in multilpliers of dual



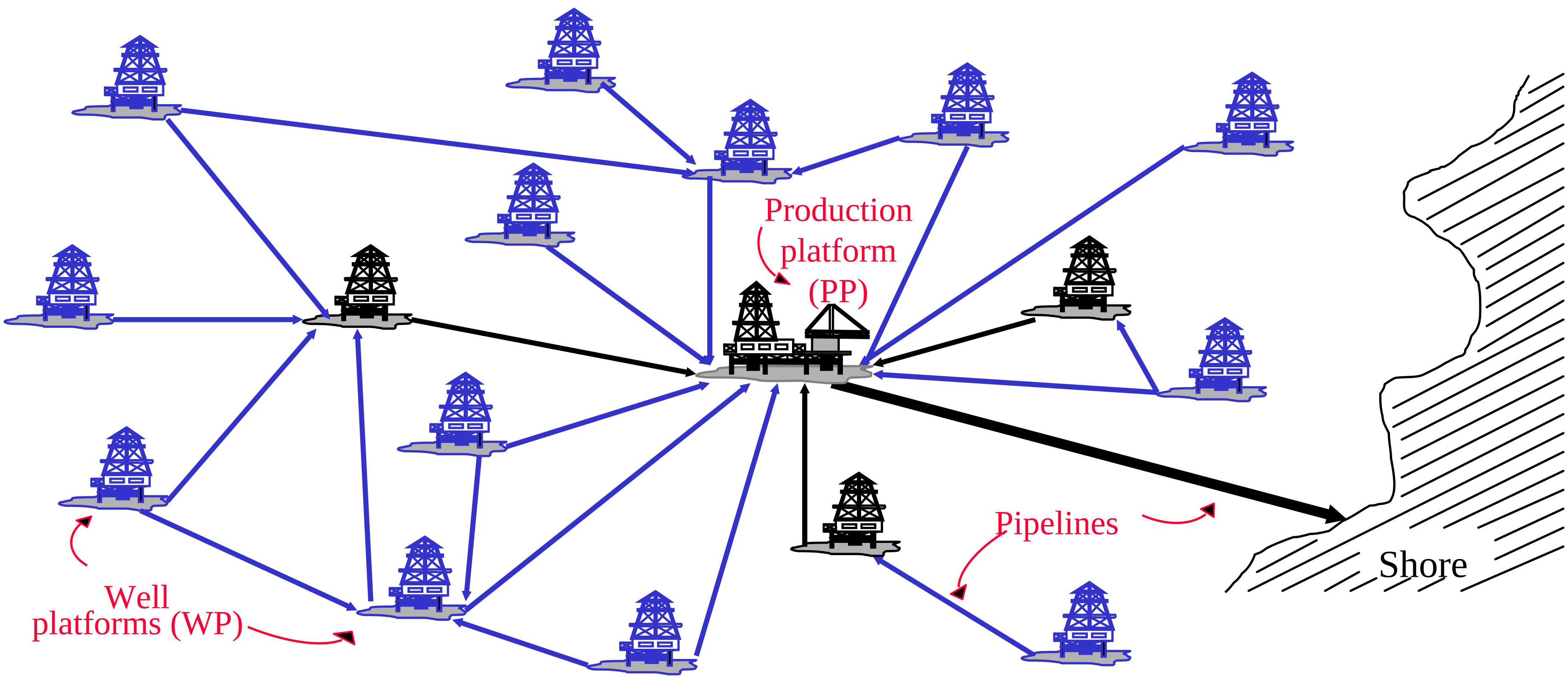
Notes: Heuristic due to dual gap
Obtaining Lower Bound might be tricky

2. Perform branch and bound search where LP relaxation is replaced by Lagrangean relaxation/decomposition to
   a) Obtain tighter bound
   b) Decompose MILP

Typically in Stochastic Programming
Caroe and Schultz (1999)
Goel and Grossmann (2006)
Tarhan and Grossmann (2008)

Remarks
1. Methods can be extended to NLP, MINLP
2. Size of dual gap depends greatly on how problems are decomposed
3. From experience gap often decreases with problem size.

Design and Planning of Offshore Oil Infrastructures *Van den Heever (2001)*



Design decisions:
WP and pipeline selections
WP type (allows connection or not)
Capacities of PP, WPs and pipelines

Planning decisions:
Investment timing for WPs, pipelines
Production profile in each time period

Objective:
Maximize NPV

Complicating factor: Complex economic objectives

Model elements (Multiperiod MINLP)

Objective Maximize Net Present Value

Reservoir and surface constraints for all t, e.g.

- Oil production vs. deliverability
- Deliverability vs. reservoir and surface pressure (nonlinear)
- Reservoir pressure vs. cumulative production (nonlinear)
- Mass balances
- Pressure balances

Logical conditions for all t, e.g.

- Each WP only invested in once
- Each WP connected to another WP or PP

Economic calculations for all t, e.g.

- Sales revenue
- Capital cost
- Operating cost
- Taxes
- Tariffs
- Royalties

Van den Heever & Grossmann (2000)

Complex economics

Solution method: Lagrangean Decomposition

Motivation: Complex economics not linked between WPs!

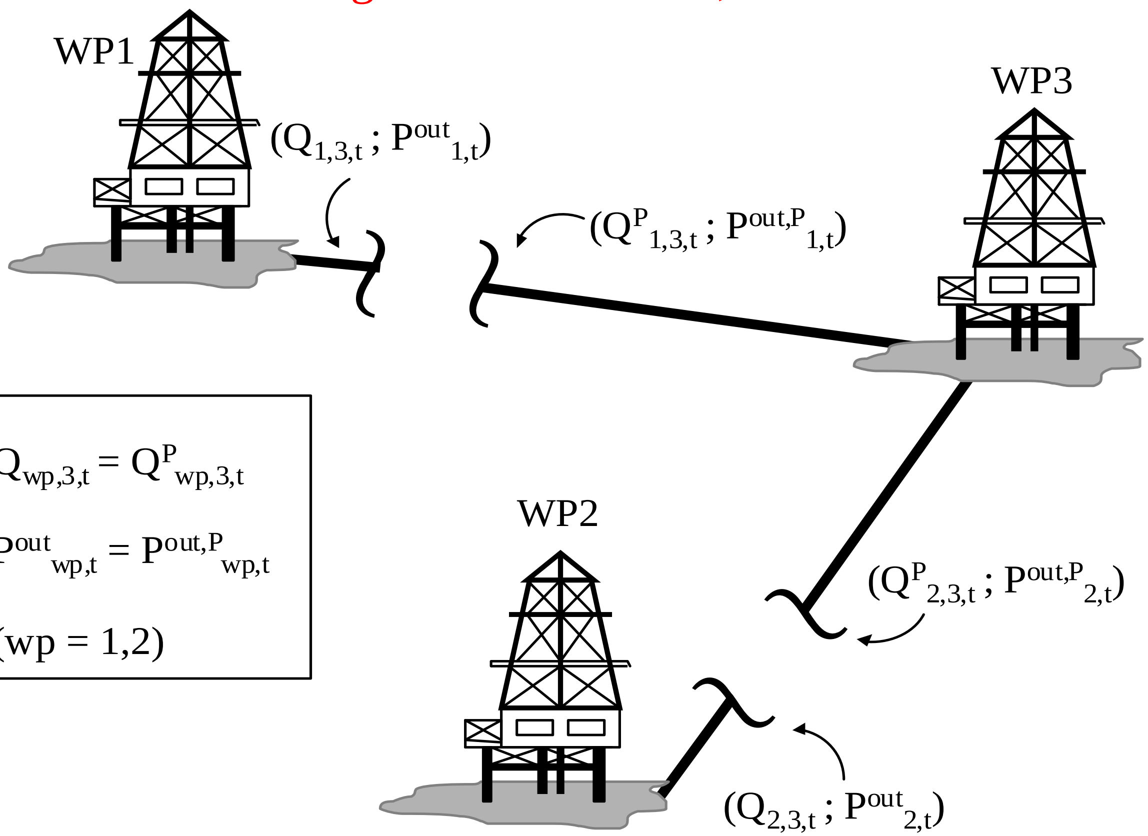
Linking variables - Flows, Pressures

Equations to dualize

$$Q_{wp,3,t} = Q^{P}_{wp,3,t}$$

$$P^{out}_{wp,t} = P^{out,P}_{wp,t}$$

$(wp = 1,2)$

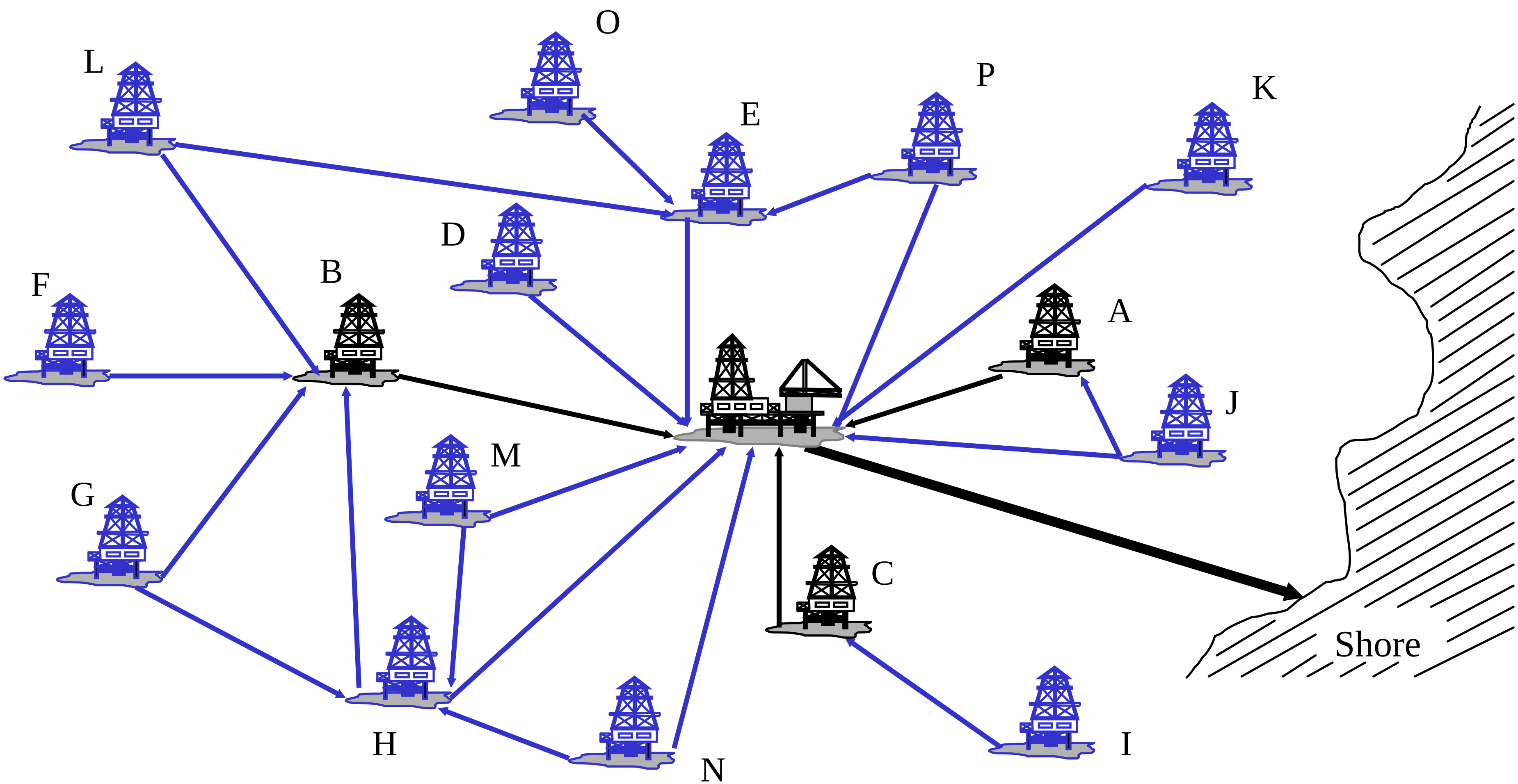


New objective = Old objective $+\sum_{wp,t}[\, \lambda^{Q}_{wp,t}\,(Q^{P}_{wp,3,t} - Q_{wp,3,t}) + \lambda^{P}_{wp,t}(P^{out,P}_{wp,t} - P^{out}_{wp,t})\,]$

Lagrangean multipliers

Model decomposes - one model each WP

Example: 16 WPs, 15 time periods



Full space yields no solution in > 5 days

Example: Results

Constraints	12696
Continuous vars.	8552
Binary vars.	759
Solution time (CPU h.)	12.3
NPV ($ mil.)	1219

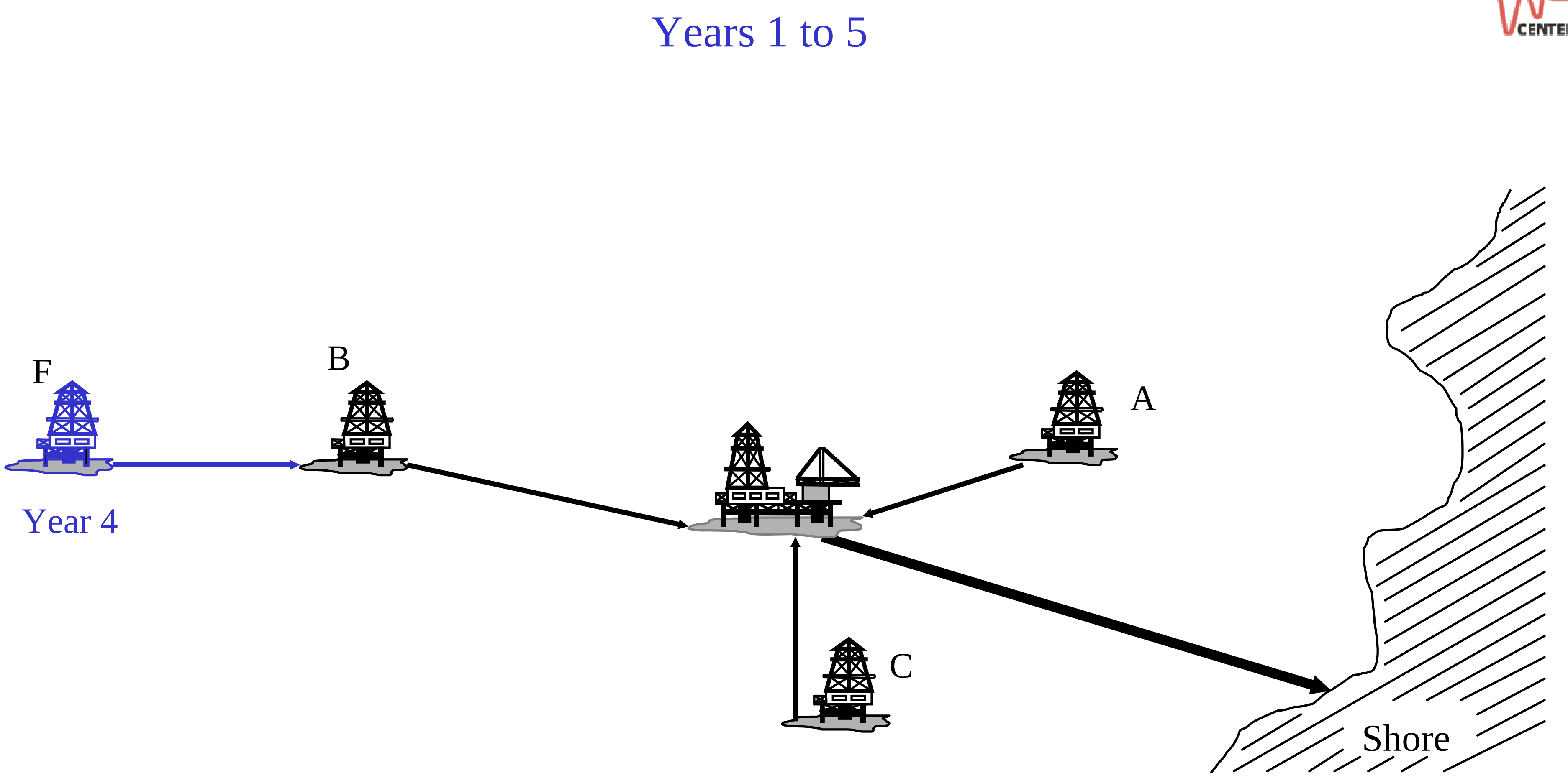
$ 95 million (8.5 %) increase in NPV compared to simple economics!

Table 9. Sequence of Bounds from the Lagrangean Decomposition Algorithm

iteration	Lagrangean soln ($ mil.)	postulated LB ($ mil.)
NLP relaxation	2271.0	
2	1611.8	*
4	1363.7	1164.4
6	1295.7	*
8	1265.2	1174.4
10	1254.8	1185.0
12	1248.6	1205.7
14	1246.2	1217.2
16	1243.5	1107.5
18	1243.3	1218.7
20	1242.8	1117.7

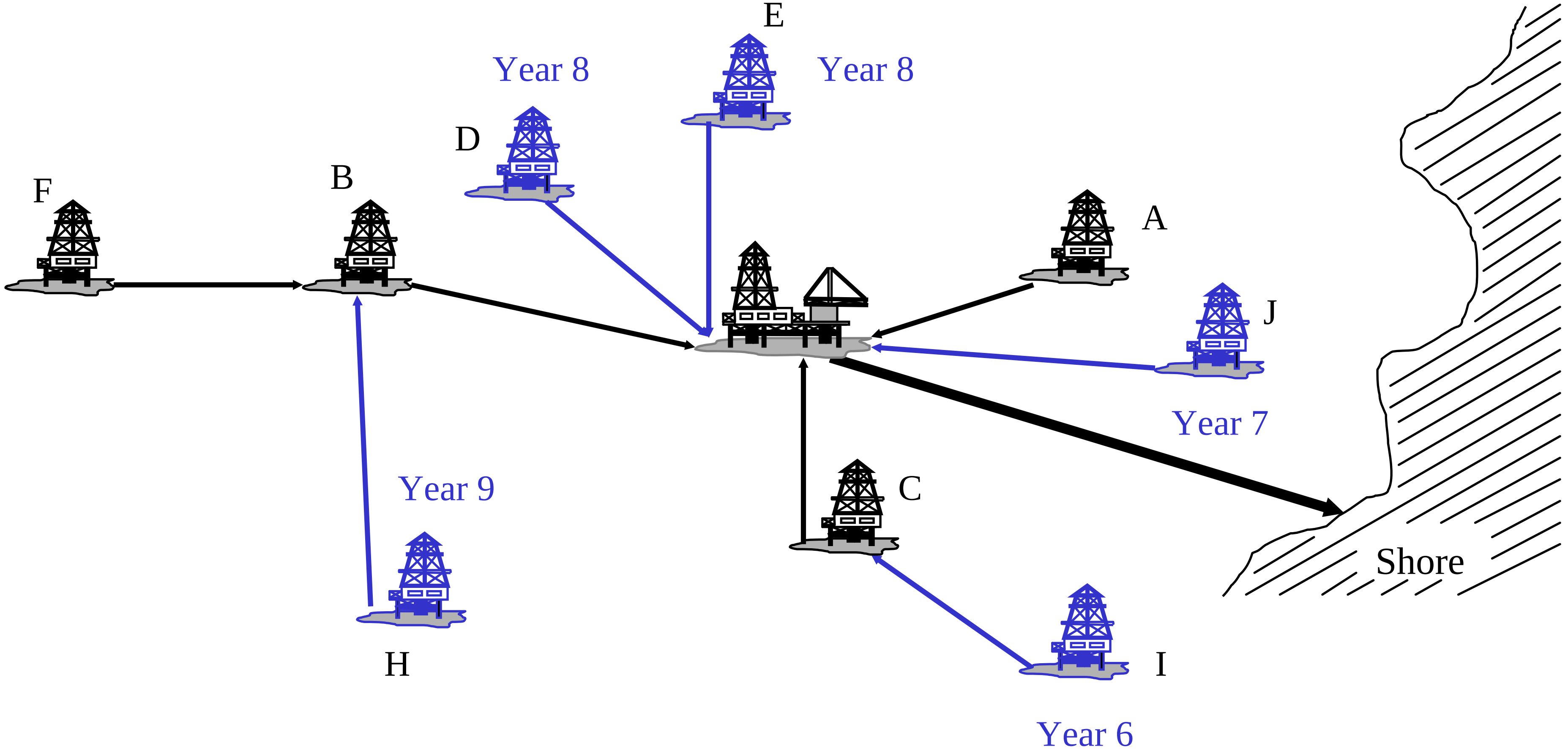
1.9% gap
after 20 iterations

Example: Solution



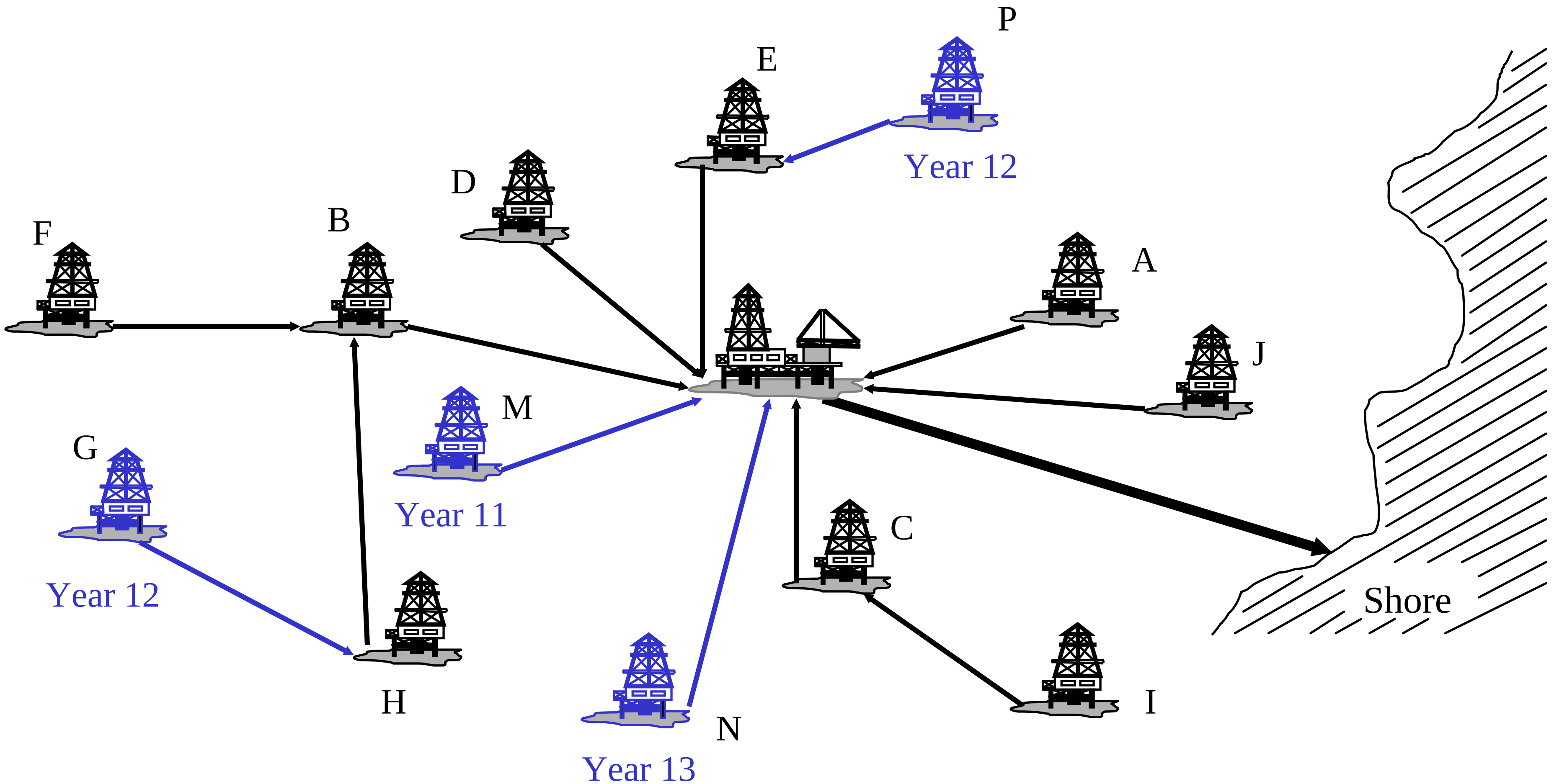
Example 2: Solution

Years 6 to 10



Example: Solution

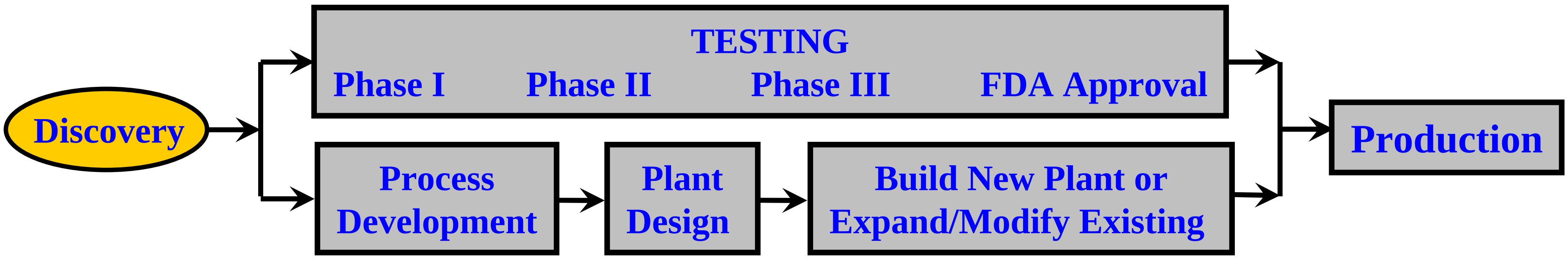
Years 11 to 15



NEW PRODUCT DEVELOPMENT

Maravelias, Grossmann (2001)

MOTIVATION:
- Increased importance of New Product Development
- Testing of Pharmaceutical and Agrochemical products
- Design and modification batch plants



DECISIONS:
1. Which products to test and how to test them
2. In what plants to invest and when
3. Production profiles

GOAL: Develop a new integration model and solution method for Simultaneous Optimization of Planning and Design Manufacturing

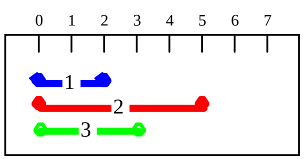
TRADE-OFFS

Testing in New Product Development

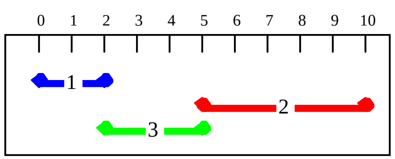
Objective: Find the schedule that maximizes the NPV
Difficulty: Probability failing tests

Three tasks to schedule

	Groundwater Studies	Acute Toxicology	Formulation Chemistry
Cost c_i:	\$200,000	\$700,000	\$100,000
Probability p_i:	0.7	0.4	0.9



Expected Value
\$1,000,000



Expected Value
\$200,000 + (0.7) \$100,000 + (0.7) (0.9) \$700,000 = \$711,000

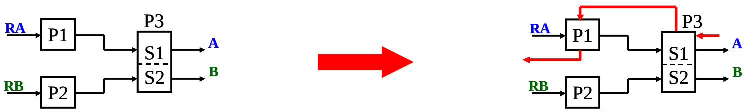
TRADE-OFF: Time to market vs cost

Design of Process Network

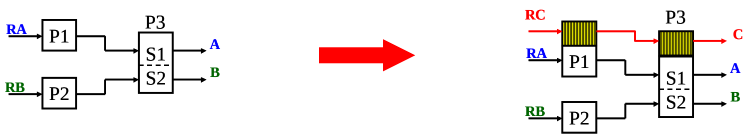
Objective: Find the optimal investment strategy
Difficulty: Risk associated with investment

DESIGN ALTERNATIVES:

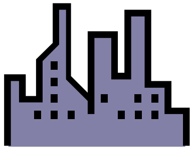
- Retrofit existing plants



- Expand existing plants



- Build new plants



TRADE-OFF: Early/large investment vs risk

PROPOSED MILP MODEL

$$Max\ NPV = Income - C_{TEST} - C_{INVEST} - C_{PURCH} - C_{OPER}$$

$$C_{INVEST} = \sum_p \sum_t \alpha N_{pt}\, yNP_{pt} + \sum_i \sum_t (\alpha E_{it}\, yE_{it} + \beta_{it}\, QE_{it})$$

$$Income = \sum_m P^m \{\sum_j \sum_l \sum_t \gamma_{jlt}\, SS_{jltm}\}$$

SCHEDULING

$s_k + d_k z_j - s_{k'} - U(w_j - y_{kk'}) \le 0$	$\forall j \in JP,\ \forall k, k' \in K(j)$
$y_{kk'} = z_j,\ y_{k'k} = 0$	$\forall j \in JP,\ \forall k, k' \in K(j),\ \forall (k,k') \in A$
$s_k + d_k w_j \le T_j$	$\forall j \in JP,\ \forall k \in K(j)$
$y_{kk'} + y_{k'k} \le w_j$	$\forall j \in JP,\ \forall k, k' \in K(j) \mid k<k'$
$\sum_{q \in (QT(k) \cap QC(r))} \hat{x}_{kq} = N_{kr}(w_j - x_k)$	$\forall j \in JP,\ \forall k \in K(j),\ \forall r \in R$
$\hat{x}_{kq} + \hat{x}_{k'q} - \hat{y}_{kk'} - \hat{y}_{k'k} \le 1$	$\forall q, \forall k \in K(q),\ \forall k' \in (K(q) \backslash KK(k)) \mid k<k'$

DESIGN & PRODUCTION PLANNING

$yE_{it} QE^L_{it} \le QE_{it} \le yE_{it} QE^U_{it}$	$\forall i \in I,\ \forall t \in T$
$Q_{it} = Q_{it-1} + QE_{it}$	$\forall i \in I,\ \forall t \in T$
$yE_{it} \le \sum_{t' \le t} yNP_{pt'}$	$\forall p \in PN,\ \forall i \in I(p),\ \forall t \in T$
$\sum_l PP_{jltm} + \sum_{i \in Oj(j)} W_{ijtm} = \sum_l SS_{jltm} + \sum_{i \in Ij(j)} W_{ijtm}$	$\forall j \in J, \forall t \in T, \forall m \in M$
$W_{ijtm} = \sum_{s \in PS(i)} \mu_{ijs}\, \rho_{is}\, \theta_{istm}$	$\forall i \in I, \forall j \in J(i,s), \forall t \in T, \forall m \in M$
$\sum_{s \in PS(i)} \theta_{istm} \le Q_{it}\, HP_{it}$	$\forall i \in I, \forall t \in T, \forall m \in M$

LINKING CONSTRAINTS

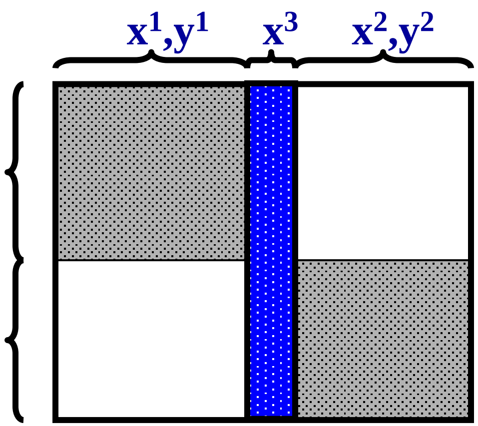
$\sum_t z_{jt} = w_j$	$\forall j \in JP$
$T_j = \sum_t T_j^t$	$\forall j \in JP$
$HT_{t-1} z_{jt} \le T_j^t \le HT_t z_{jt}$	$\forall j \in JP, \forall t \in T$
$S_{jltm} \le d_{jlt}^U f_{mj} H_t \sum_{t' \le t} z_{jt'}$	$\forall j \in JP,\ \forall l \in L,\ \forall t \in T,\ \forall m \in M$
$S_{jltm} \le (HT_t - T_j^t)\, d_{jlt}^U f_{mj}$	$\forall j \in JP,\ \forall l \in L,\ \forall t \in T,\ \forall m \in M$

SOLUTION METHOD

- Constraint matrix special structure that can be exploited

(I) Testing & Linking

(II) Design/ Planning



x^1, x^2, x^3 : Vectors of continuous variables
y^1, y^2 : Vectors of discrete variables

- Use **Lagrangean Decomposition** to get two independent problems
  - (Fisher, 1985; Guignard & Kim, 1987)
- Solve two decoupled problems separately (computationally cheaper)
- Combine independent solutions to obtain solution of integrated problem

Formulation Decomposed MILP

$$(P)\quad Max\ NPV = c^1x^1 + c^2x^2 + c^3x^3 + dy^2$$
$$s.t.\quad A^1x^1 + A^{31}x^3 + B^1y^1 \le b^1 \qquad (I)$$
$$A^2x^2 + A^{32}x^3 + B^2y^2 \le b^2 \qquad (II)$$

$$(P')\quad Max\ NPV = c^1x^1 + c^2x^2 + c^3x^3 + dy^2 - \lambda(x^{32} - x^{31})$$
$$s.t.\quad A^1x^1 + A^{31}x^{31} + B^1y^1 \le b^1 \qquad (I)$$
$$A^2x^2 + A^{32}x^{32} + B^2y^2 \le b^2 \qquad (II)$$
$$\cancel{x^{31} = x^{32}} \qquad (III)$$

Scheduling Subproblem

$$(P1)\quad Max\ NPV1 = c^1x^1 + \lambda x^{31}$$
$$s.t.\quad A^1x^1 + A^{31}x^{31} + B^1y^1 \le b^1 \qquad (I)$$

Design/Planning Subproblem

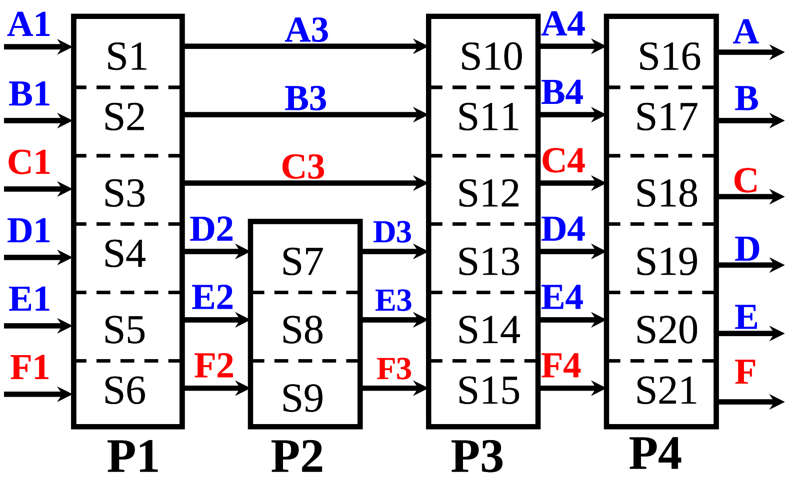
$$(P2)\quad Max\ NPV2 = c^2x^2 + c^3x^{32} + dy^2 - \lambda x^{32}$$
$$s.t.\quad A^2x^2 + A^{32}x^{32} + B^2y^2 \le b^2 \qquad (II)$$

λ: Lagrange Multipliers

EXAMPLE: Process Network

Existing Proteins: A, B, D, E

Potentially New Proteins: C, F



Demand forecasts (kg/month)

Product	1st yr	2nd yr	3rd yr	4th yr	5th yr	6th yr
A	3	3	3	3	3	3
B	3	3	3	4	4	4
C	4	4	4	4	4	4
D	5	5	5	5	5	5
E	4	4	2	2	0	0
F	6	6	6	6	6	6

Prices of Chemicals ($\$10^3$/kg)

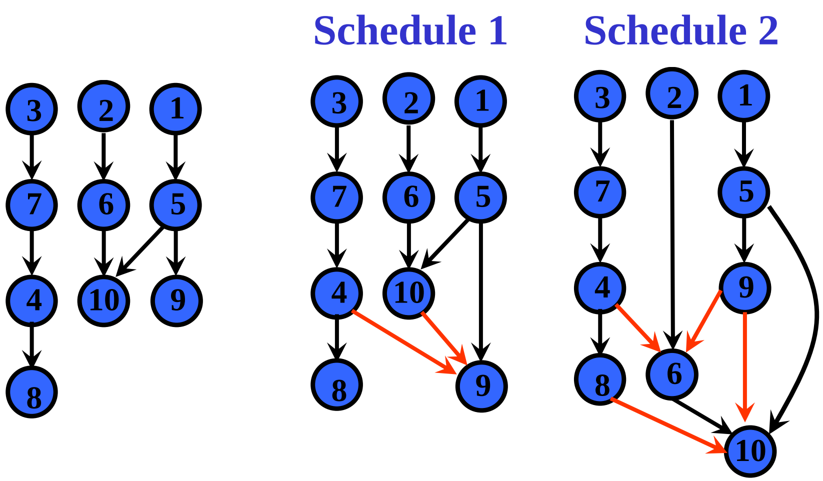
Raw Material	Price	Product	Price
A1	40	A	300
B1	50	B	350
C1	60	C	550
D1	50	D	450
E1	60	E	400
F1	70	F	700

Process Design Data

Process	Conversion	Capacity (kg/month)	Fixed ($\$10^3$)	Variable ($\$10^3 \cdot$month/kg)	Operating ($\$10^3$/kg)
P1	0.77	10	4,500	1,100	18
P2	0.83	6	4,000	800	14
P3	0.71	8	3,500	1,000	16
P4	0.91	8	5,000	1,200	14

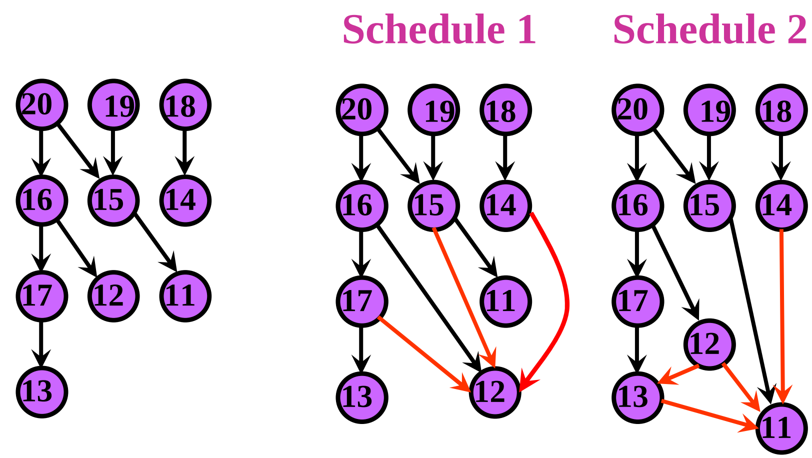
EXAMPLE: Testing Tasks

Product C



Test	Cost ($\$10^4$)	Cost of out-sourcing ($\$10^4$)	Duration (months)	Prob/lity of success	Resource Req/ment
1	80	160	5	1	Lab1
2	80	160	4	1	Lab2
3	50	100	4	1	Lab3
4	10	20	1	0.84	Lab4
5	490	980	3	0.98	Lab1
6	111	222	6	1	Lab2
7	60	120	1	0.95	Lab3
8	1740	3480	7	1	Lab4
9	620	1240	9	1	Lab1
10	10	20	1	1	Lab2

Product F



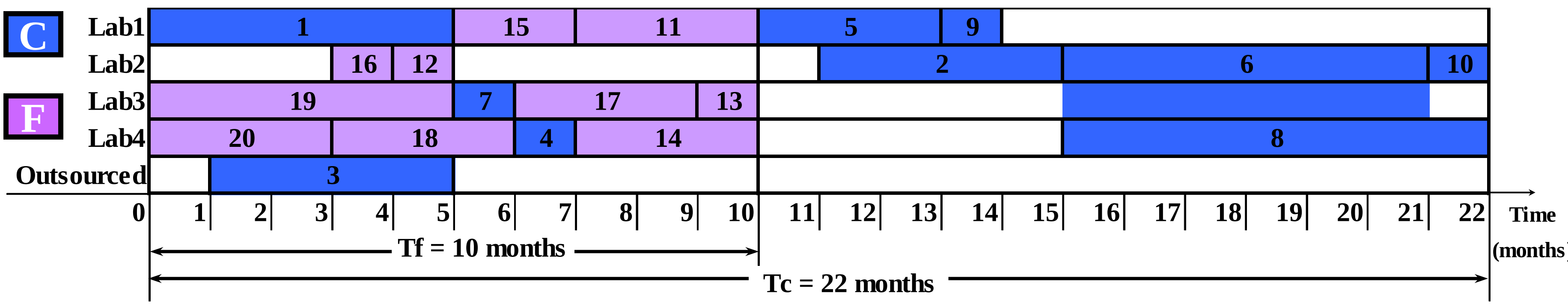
Test	Cost ($\$10^4$)	Cost of out-sourcing ($\$10^4$)	Duration (months)	Prob/lity of success	Resource Req/ment
11	160	320	3	1	Lab1
12	1130	2260	1	0.87	Lab2
13	10	20	1	0.91	Lab3
14	130	260	3	1	Lab4
15	530	1060	2	1	Lab1
16	90	180	1	1	Lab2
17	117	234	3	1	Lab3
18	400	800	3	1	Lab4
19	570	1140	5	1	Lab3
20	230	460	3	1	Lab4

Solution Scheduling Tests & Design Batch Plant

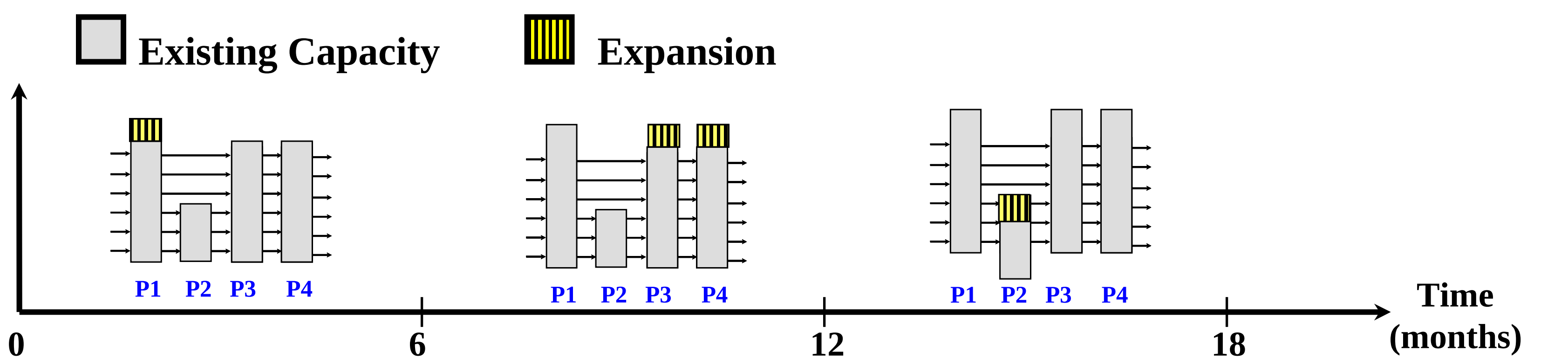
HEURISTIC: Min Completion Time $\Rightarrow T_c = 10, T_F = 17,$ ***NPV = \$7.62 billion***

PRPOSED MILP MODEL: $T_c = 10, T_F = 22,$ ***NPV = \$8.12 billion (+7%)***

TESTING: Gantt Charts for Resources

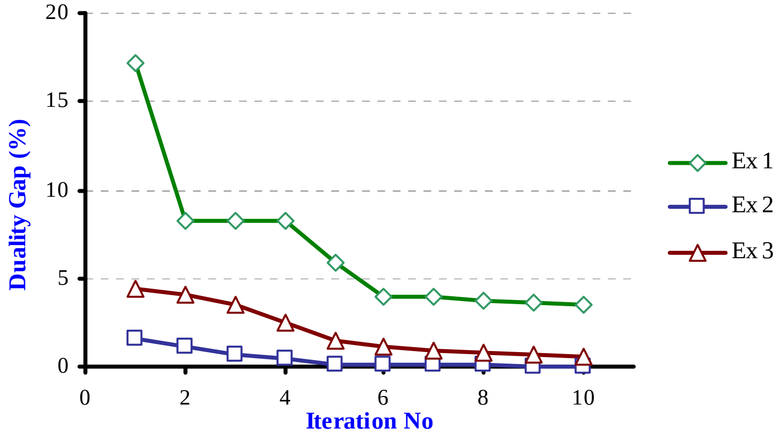


DESIGN: Expansions of Processes



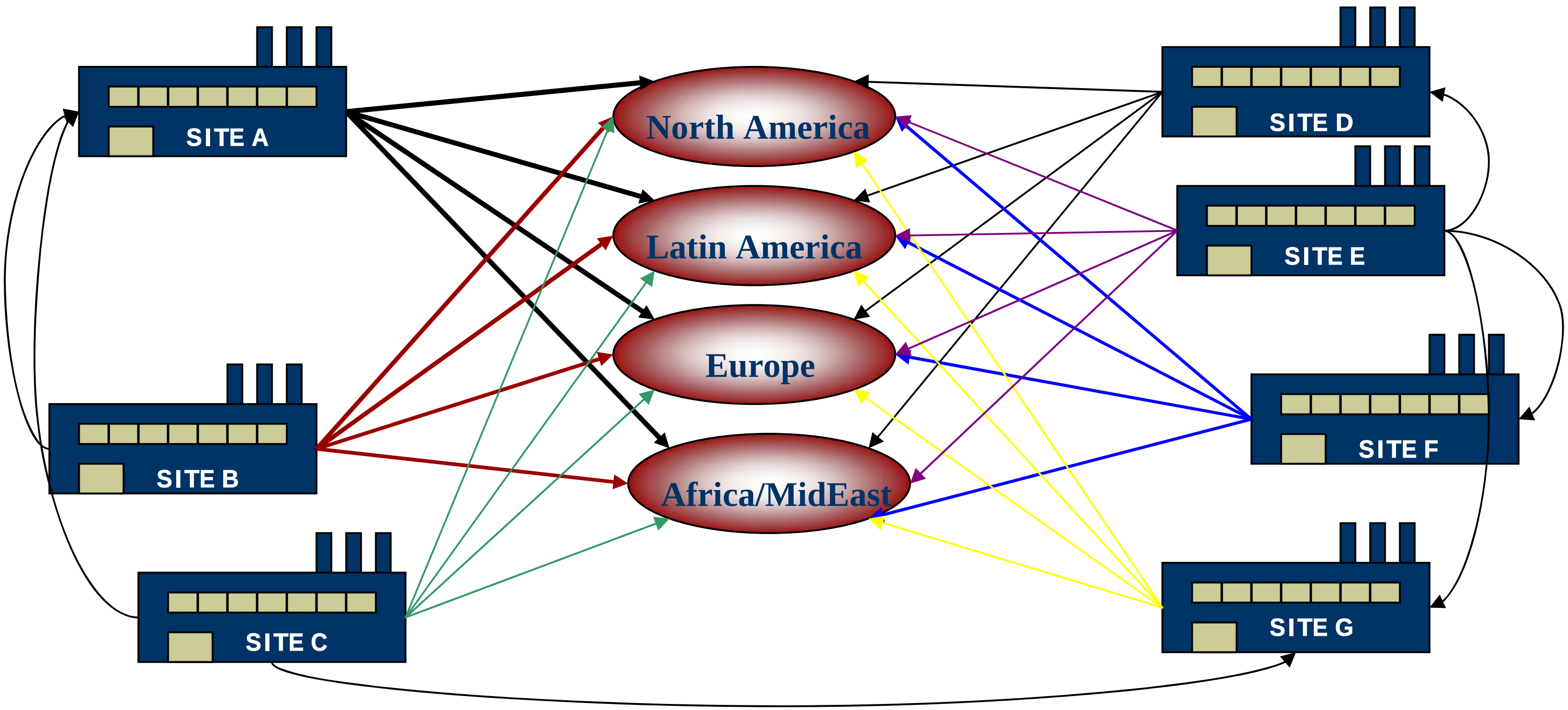
COMPUTATIONAL RESULTS

	Example 1	Example 2	Example 3
Binary Variables	236	354	612
Continuous Variables	9,372	7,456	32,184
Constraints	8,255	6,825	30,903
Full Space Method			
Optimal Solution	9,518	94,986	135,024
CPU Time (sec)	8.9	57.2	836.6
Decomposition Heuristic			
Best Solution	9,518	94,986	134,834
Major iterations	5	3	6
CPU Time (sec)	20.5	37.6	252.4
Relative Duality Gap	3.96%	0.10%	0.77%



Multisite Distribution Network

Jackson, Grossmann,Wassick, Hoffman (2002)



- **Objective: Develop model and effective solution strategy for large-scale multiperiod planning with *Nonlinear Process Models***

Multisite Distribution Model

•Develop Multisite Model to determine:

1)What products to manufacture in each site

2)What sites will supply the products for each market

3)Production and inventory plan for each site

➢***Objective: Maximize Net Present Value***

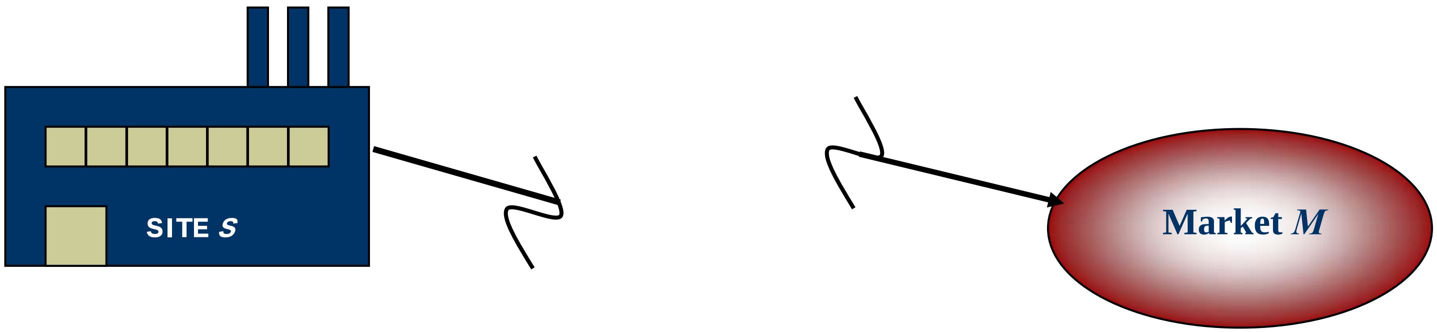
•**Challenges/Optimization Bottlenecks: Large-Scale NLP**

–Interconnections between time periods & sites/markets

➢*Apply* ***Lagrangean Decompostion*** *Method*

Spatial Decomposition

$$\max \ PROFIT = SCost_S^{PR,M} * SALES_S^{PR,M} - PCost_S^{PR,M} * PROD_S^{PR,M} \boxed{+ \lambda_S^{PR,M}\left(PROD_S^{PR,M} - SALES_S^{PR,M}\right)}$$



SITE S CONSTRAINTS :

$$f\left(PROD_S^{PR,M}\right) \le 0$$

$$\max\left(- PCost_S^{PR,M}\, PROD_S^{PR,M} \boxed{+ \lambda_S^{PR,M}\, PROD_S^{PR,M}}\right)$$

Site SUBPROBLEM for all *S* *(NLP)*

Market M CONSTRAINTS :

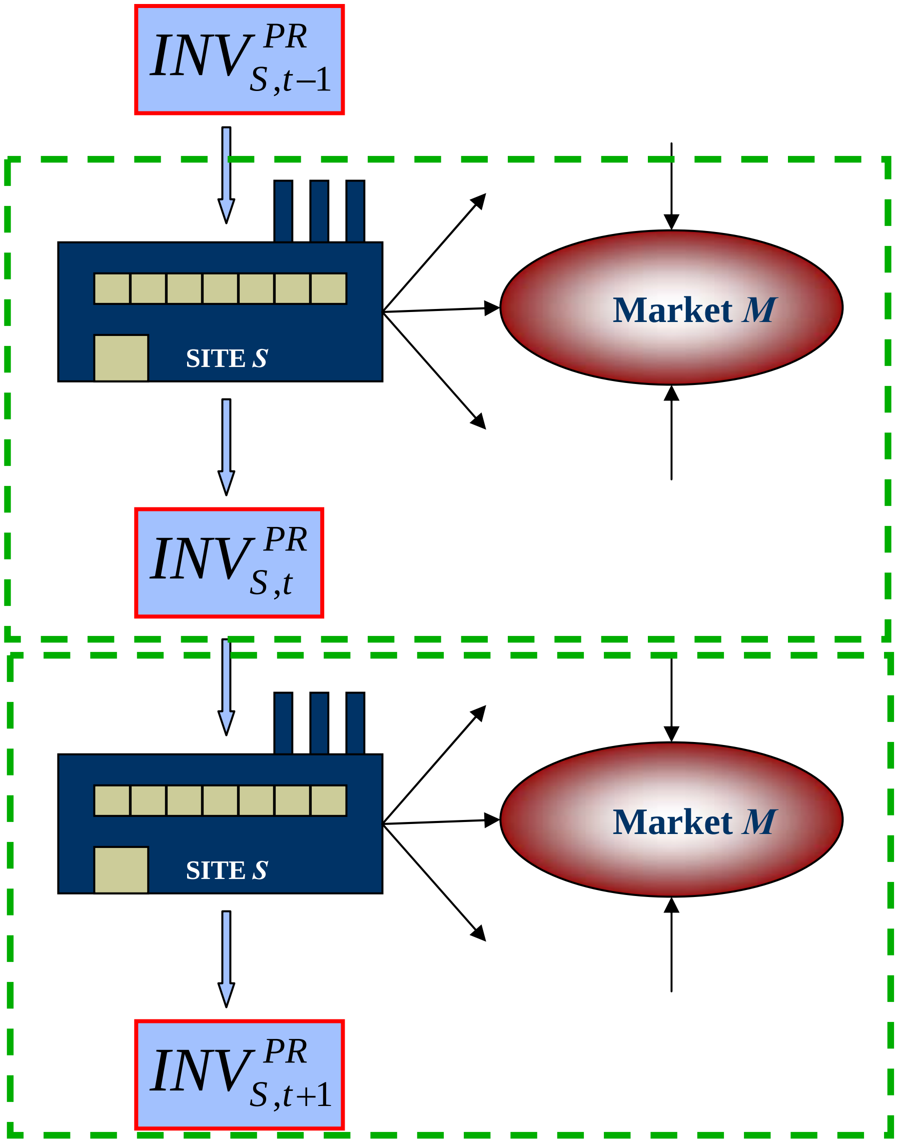
$$f\left(SALES_S^{PR,M}\right) \le 0$$

$$\max\left(SCost_S^{PR,M}\, SALES_S^{PR,M} \boxed{- \lambda_S^{PR,M}\, SALES_S^{PR,M}}\right)$$

Market SUBPROBLEM for all *M* *(LP)*

Temporal Decomposition

- Decompose at each time period
- Duplicate variables for Inventories for each time period
- Apply Langrangean Decomposition Algorithm



Multisite Distribution Model - Spatial

- 3 Multi-Plant Sites, 3 Geographic Markets
- Solved with GAMS/Conopt2

# Time Periods (months)	Variables/ Constraints	Optimal Solution Profit (million-$)	Full Space Solution Time (CPU sec)	Lagrangean Solution Time (CPU sec)	% Within Full Optimal Solution
2	3345 / 2848	164	52	10	10%
4	6689 / 5698	326	478	127	11%
6	10033 /8548	497	1605	279	9%
8	13377/11398	666	2350	550	9%

Multisite Distribution Model - Temporal

- 3 Multi-Plant Sites, 3 Geographic Markets
- Solved with GAMS/Conopt2

# Time Periods (months)	Variables/ Constraints	Optimal Solution Profit (million-$)	Full Space Solution Time (CPU sec)	Lagrangean Solution Time (CPU sec)	% Within Full Optimal Solution
3	5230 / 5005	116.05	395	97	2.2
6	9973 / 8551	236.53	2013	138	2.3
12	19945 /17101	474.18	10254	278	2.2

Temporal much smaller gap!

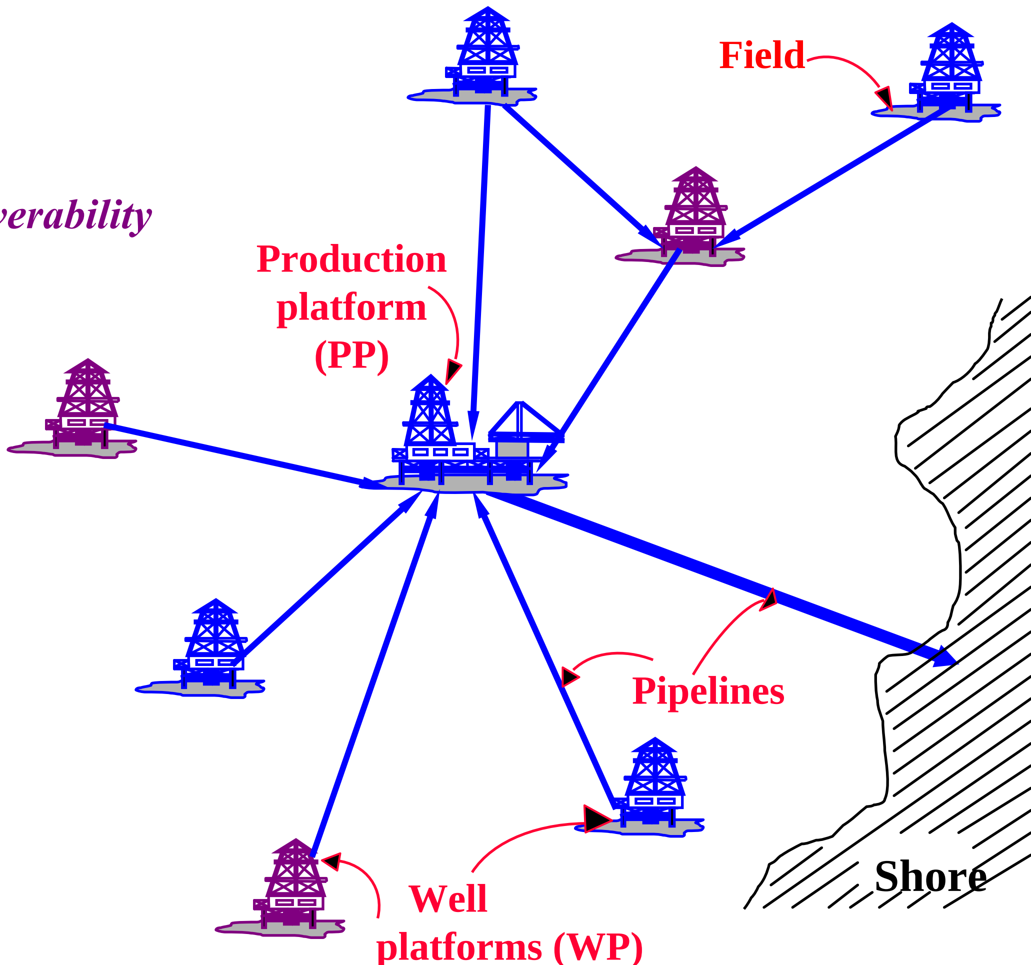
Reason: material balances not violated at each time period

Stochastic Optimization for Gasfield Planning

Goel , Grossmann, El-Amr, Mulckay (2006)

Superstructure

- Fields → **Certain Fields**
- Fields → **Uncertain Fields** *Size* *Deliverability*
- Well Platforms (WP)
- Connecting pipelines
- Production Platforms (PP)



Decisions

•Investment

–Platforms: Which and When

–Pipelines: Which and When

–Platform Capacities

•Operational

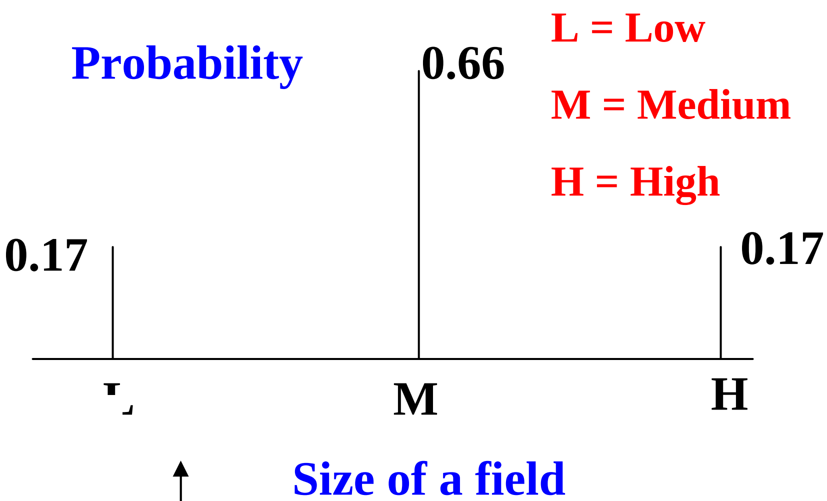
–Production profile: each field

Time Horizon

Discretized into time periods: 1 year each

Representation of Uncertainty

- **Discrete probability distribution**
  - **Size and Initial Deliverability of each uncertain field**



•Scenario

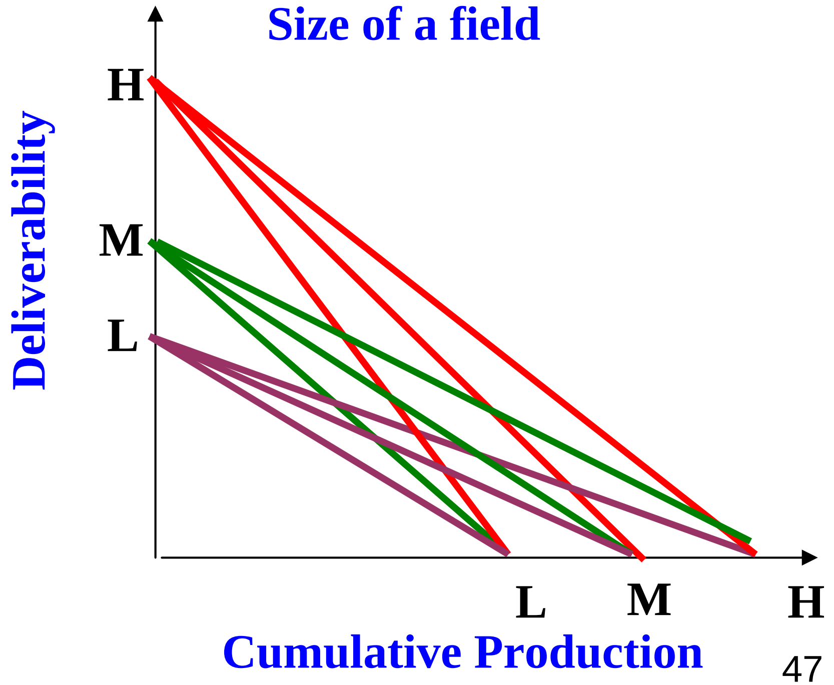
–Possible combination of sizes and initial deliverabilities of different uncertain fields

–Each scenario has a given probability

•Objective

–Maximize Expected NPV

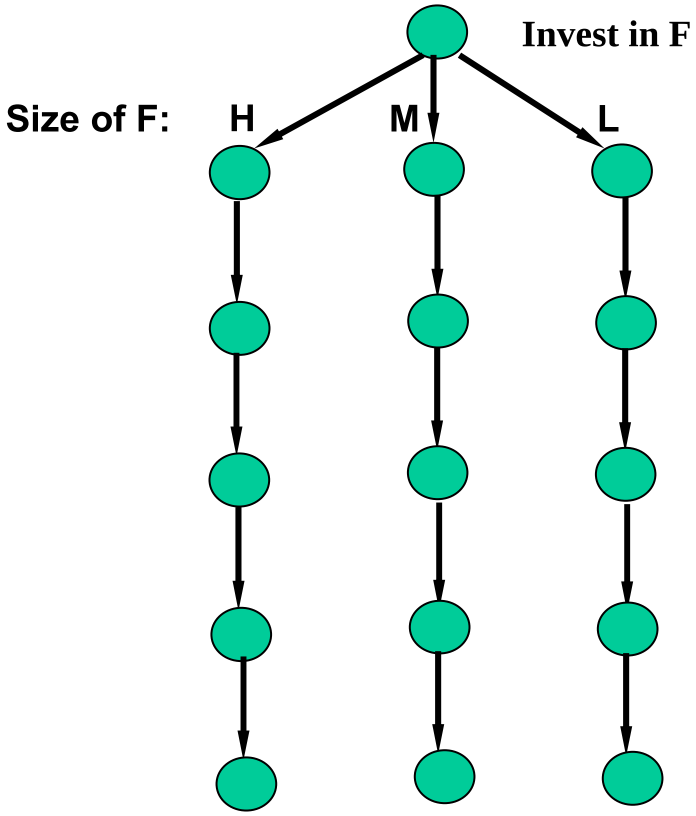
–Expected NPV (ENPV) = $\sum_s p^s NPV^s$



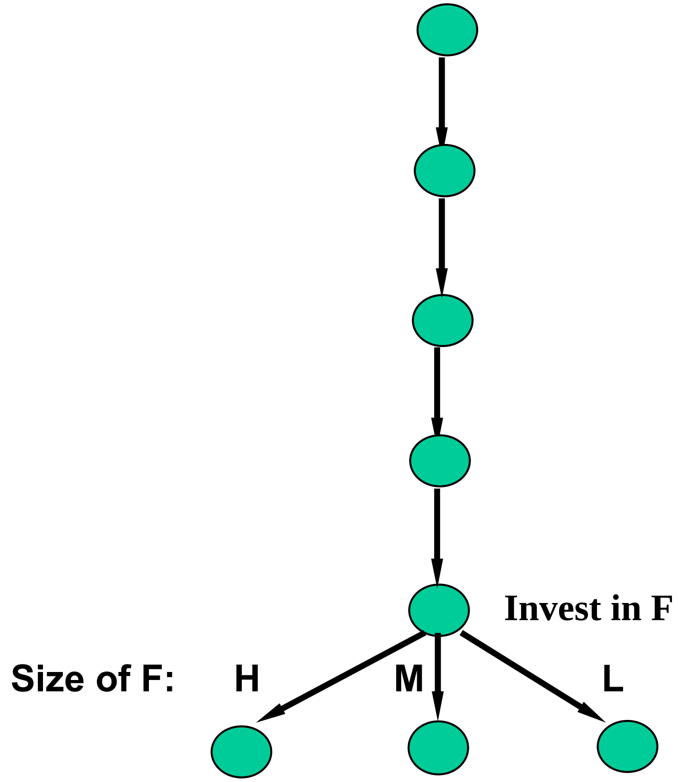
Decision Dependent Scenario Trees

Assumption: **Uncertainty in a field resolved as soon as WP installed at field**

Invest in F in year 1



Invest in F in year 5



Scenario tree

- **Not unique: Depends on timing of investment at uncertain fields**
- **Central to defining a Stochastic Programming Model**

Novel Stochastic Programming Model (SPM)

Maximize Expected NPV

$$Max \quad \sum_s p^s \quad [\sum_t (c_{1t} q_t^s + c_{2t} d_t^s + c_{3t} y_t^s + \sum_{uf} c_{4t,uf} b_{uf,t}^s)]$$

Linear

Investment, Operation Constraints

$$A^s q_t^s \le a^s \qquad \forall (t,s)$$

$$g_t(q_1^s, q_2^s, \ldots, q_t^s) \le 0 \qquad \forall (t,s)$$

$$h_t(d_1^s, d_2^s, \ldots, d_t^s) \le 0 \qquad \forall (t,s)$$

$$\sum_t b_{uf,t}^s \le 1 \qquad \forall (uf, s)$$

$$r_{uf,t}(q_t^s, d_t^s, b_{uf,1}^s, b_{uf,2}^s, \ldots, b_{uf,t}^s, y_1^s, y_2^s, \ldots, y_t^s) \le 0 \qquad \forall (uf, t, s)$$

q_t^s: Cont. Operation vars.
d_t^s: Cont. Investment vars.
$b^s_{uf,t}$: 0-1 Investment vars., uf
y^s_t: Other 0-1 investment vars.
$Z_t^{s,s'}$: Boolean variables

Non-Anticipativity

Constraints

uf: uncertain field
s, s': scenario
t: time period

Novel Stochastic Programming Model (SPM)

Maximize Expected NPV

$$Max \quad \sum_s p^s \quad [\sum_t (c_{1t} q_t^s + c_{2t} d_t^s + c_{3t} y_t^s + \sum_{uf} c_{4t,uf} b_{uf,t}^s)]$$

Linear Investment, Operation Constraints

$$A^s q_t^s \leq a^s \qquad \forall (t,s)$$

$$g_t(q_1^s, q_2^s, \ldots, q_t^s) \leq 0 \qquad \forall (t,s)$$

$$h_t(d_1^s, d_2^s, \ldots, d_t^s) \leq 0 \qquad \forall (t,s)$$

$$\sum_t b_{uf,t}^s \leq 1 \qquad \forall (uf, s)$$

$$r_{uf,t}(q_t^s, d_t^s, b_{uf,1}^s, b_{uf,2}^s, \ldots, b_{uf,t}^s, y_1^s, y_2^s, \ldots, y_t^s) \leq 0 \qquad \forall (uf, t, s)$$

q_t^s: Cont. Operation vars.

d_t^s: Cont. Investment vars.

$b^s_{uf,t}$: 0-1 Investment vars., uf

y^s_t: Other 0-1 investment vars.

$Z_t^{s,s'}$: Boolean variables

Non-Anticipativity Constraints

$$\begin{bmatrix} Z_t^{s,s'} \\ q_t^s = q_t^{s'} \\ d_{t+1}^s = d_{t+1}^{s'} \\ y_{t+1}^s = y_{t+1}^{s'} \\ b_{uf,t+1}^s = b_{uf,t+1}^{s'} \quad \forall uf \end{bmatrix} \bigvee \begin{bmatrix} \neg Z_t^{s,s'} \\ q_t^s, q_t^{s'} \geq 0 \\ d_{t+1}^s, d_{t+1}^{s'} \geq 0 \\ y_{t+1}^s, y_{t+1}^{s'} \in \{0,1\}^{\dim(y)} \\ b_{uf,t+1}^s, b_{uf,t+1}^{s'} \in \{0,1\} \end{bmatrix} \qquad \forall (t, s, s')$$

$$Z_t^{s,s'} \Leftrightarrow \bigwedge_{uf \in D(s,s')} [\bigwedge_{\tau=1}^{t} (\neg b_{uf,\tau}^s)] \qquad \forall (t, s, s')$$

$$b_{uf,1}^s = b_{uf,1}^{s'} \qquad \forall (s, s')$$

$$y_1^s = y_1^{s'} \qquad \forall (s, s')$$

$$d_1^s = d_1^{s'} \qquad \forall (s, s')$$

uf: uncertain field

s, s': scenario

t: time period

Generalized Disjunctive Program

Branch and Bound Algorithm

Major steps (Maximization problem) at every node

- **Upper bound: Lagrangean dual**
- **Lower bound: Feasible solutions**
- **Branching**

Formulation of Lagrangean dual

Relaxation

- **Relax disjunctions, logic constraints**
- **Penalty for equality constraints**

$^{b}\lambda_{uf}^{s,s'}, {}^{y}\lambda^{s,s'}, {}^{d}\lambda^{s,s'}$: Lagrange Multipliers

$$\text{Max} \sum_s p^s \left[\sum_t \left(c_{1t} q_t^s + c_{2t} d_t^s + c_{3t} y_t^s + \sum_{uf} c_{4t,uf} b_{uf,t}^s \right) \right]$$
$$+ \sum_{(s,s')} \left[\sum_{uf} {}^{b}\lambda_{uf}^{s,s'} \left(b_{uf,1}^s - b_{uf,1}^{s'} \right) + {}^{y}\lambda^{s,s'} \left(y_1^s - y_1^{s'} \right) + {}^{d}\lambda^{s,s'} \left(d_1^s - d_1^{s'} \right) \right]$$

$$\sum_{\tau=1}^{t} \left(A_\tau^s q_\tau^s + B_\tau^s d_\tau^s + C_\tau^s y_\tau^s + \sum_{uf} D_{uf,\tau}^s b_{uf,\tau}^s \right) \le a_t^s \qquad \forall (t,s)$$

$$\begin{bmatrix} & Z_t^{s,s'} & \\ q_t^s & = & q_t^{s'} \\ d_{t+1}^s & = & d_{t+1}^{s'} \\ y_{t+1}^s & = & y_{t+1}^{s'} \\ b_{uf,t+1}^s & = & b_{uf,t+1}^{s'} \quad \forall uf \end{bmatrix} \vee \left[\neg Z_t^{s,s'} \right] \qquad \forall (t,s,s')$$

$$Z_t^{s,s'} \Leftrightarrow \bigwedge_{uf \in \mathcal{D}(s,s')} \left[\bigwedge_{\tau=1}^{t} \left(\neg b_{uf,\tau}^s \right) \right] \qquad \forall (t,s,s')$$

$$b_{uf,1}^s = b_{uf,1}^{s'} \qquad \forall (uf,s,s')$$
$$d_1^s = d_1^{s'} \qquad \forall (s,s')$$
$$y_1^s = y_1^{s'} \qquad \forall (s,s')$$

Formulation of Lagrangean dual

Relaxation

- **Relax disjunctions, logic constraints**
- **Penalty for equality constraints**

${}^{b}\lambda_{uf}^{s,s'}, {}^{y}\lambda^{s,s'}, {}^{d}\lambda^{s,s'}$: Lagrange Multipliers

$$\phi(\lambda) = \text{Max} \sum_s p^s \left[\sum_t \left(c_{1t} q_t^s + c_{2t} d_t^s + c_{3t} y_t^s + \sum_{uf} c_{4t,uf} b_{uf,t}^s \right) \right] + \sum_{(s,s')} \left[\sum_{uf} {}^{b}\lambda_{uf}^{s,s'} \left(b_{uf,1}^s - b_{uf,1}^{s'} \right) + {}^{y}\lambda^{s,s'} \left(y_1^s - y_1^{s'} \right) + {}^{d}\lambda^{s,s'} \left(d_1^s - d_1^{s'} \right) \right]$$

$$\sum_{\tau=1}^{t} \left(A_\tau^s q_\tau^s + B_\tau^s d_\tau^s + C_\tau^s y_\tau^s + \sum_{uf} D_{uf,\tau}^s b_{uf,\tau}^s \right) \leq a_t^s \quad \forall (t,s)$$

Model decomposes: one MILP, each scenario!!!

$$\phi^s(\lambda) = \text{Max} \ p^s \left[\sum_t \left(c_{1t} q_t^s + c_{2t} d_t^s + c_{3t} y_t^s + \sum_{uf} c_{4t,uf} b_{uf,t}^s \right) \right] + \sum_{s'} (-1)^{k(s')} \left[\sum_{uf} {}^{b}\lambda_{uf}^{s,s'} b_{uf,1}^s + {}^{y}\lambda^{s,s'} y_1^s + {}^{d}\lambda^{s,s'} d_1^s \right]$$

$$\sum_{\tau=1}^{t} \left(A_\tau^s q_\tau^s + B_\tau^s d_\tau^s + C_\tau^s y_\tau^s + \sum_{uf} D_{uf,\tau}^s b_{uf,\tau}^s \right) \leq a_t^s \quad \forall t$$

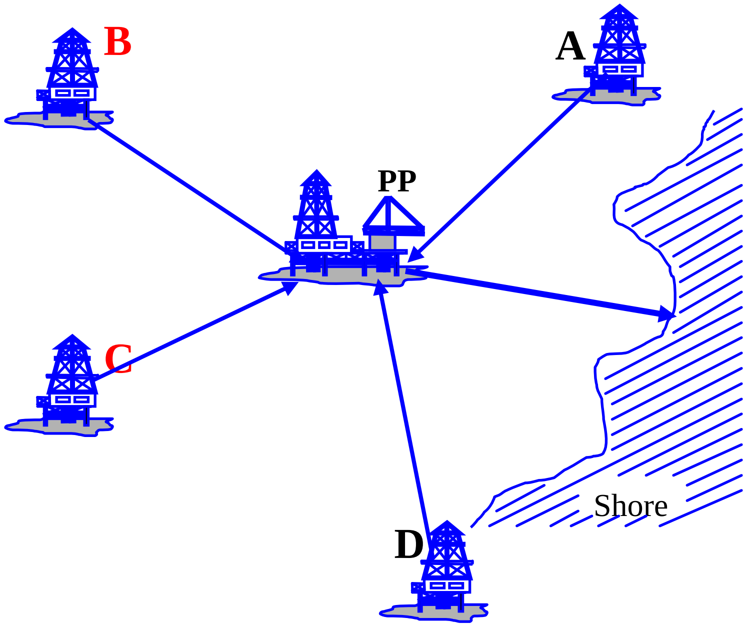
Lagrangean Relaxation: Upper bound for any λ

$$\sum_s \phi^s(\lambda) = \phi(\lambda) \geq \phi_{optimal} \quad \forall \lambda$$

Lagrangean Dual: Tightest upper bound

$$\phi_{LD} = \min_\lambda \phi(\lambda) \geq \phi_{optimal}$$

Example



15 time periods

Uncertainty sizes of fields B and C: 9 scenarios

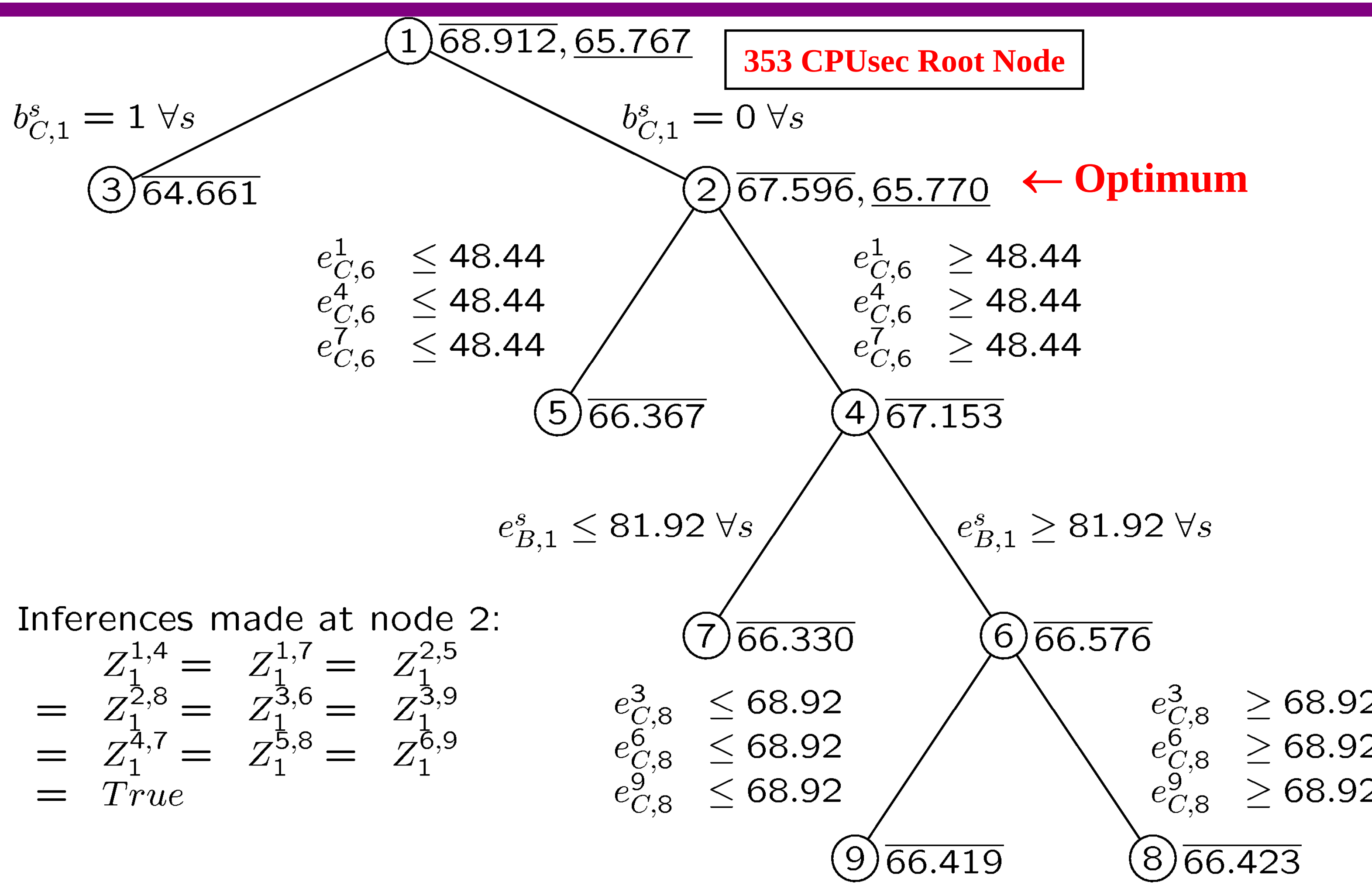
> **Size MILP Model:**
> **16,281 0-1, 125,956 cont var, 386,597 const**

Stochastic programming solution

	Scenario								
	1	4	7	2	5	8	3	6	9
$t = 1$	A(66.68), B(83.32), PP(150.00)								
$t = 6$	C(73.86)								
$t = 7$				C(73.33)					
$t = 8$							C(73.33)		
$t = 10$	D(66.26)								
$t = 12$		D(69.75)		D(70.54)					
$t = 13$			D(49.01)		D(55.18)		D(60.99)		
ENPV	$65.770 Million								

$65.770 Million Stochastic vs Deterministic solution: $61.215 Million

Branch and Bound Tree



Inferences made at node 2:

$$
\begin{aligned}
& Z_1^{1,4} = Z_1^{1,7} = Z_1^{2,5} \\
= \; & Z_1^{2,8} = Z_1^{3,6} = Z_1^{3,9} \\
= \; & Z_1^{4,7} = Z_1^{5,8} = Z_1^{6,9} \\
= \; & True
\end{aligned}
$$

9 nodes, 1683 CPU sec (1% optimality gap)

Concluding Remarks

1. **Lagrangean relaxation/decomposition is well established method for solving large-scale MILPs, MINLPs, NLPs**

2. **Non-obvious part is how to apply it to effectively decompose a problem**

Some guidelines:

a) **Use Lagragean decomposition, not Lagrangean relaxation**
a) **Avoid if possible relaxing "critical" constraints (eg mass balances)**
b) **Try to decompose so that effect of relaxed equations is not too large**