Improving Dual Bound for Stochastic MILP Models Using Sensitivity Analysis

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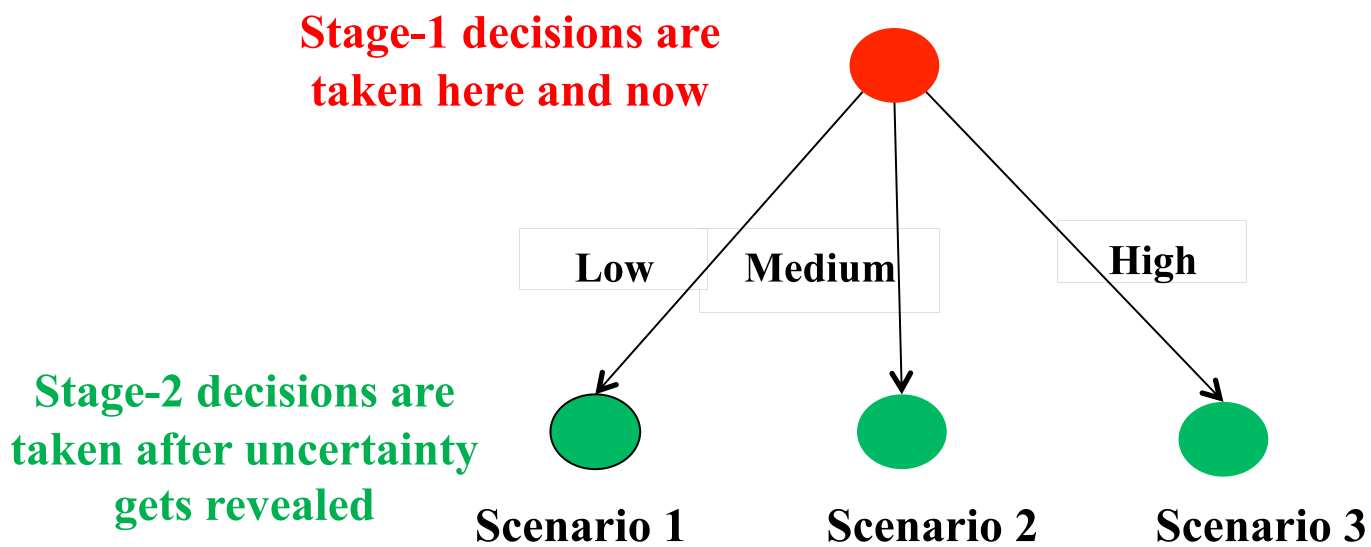
Stochastic Programming

Why we use stochastic programming?

- To model those problems where some of the parameters are **random**.
  (e.g. uncertain **Reservoir Size, Demand, Prices**)
- Taking **probability distribution** into account while making decisions and having possibility of corrective actions in the future **(recourse)**

The main idea behind two-stage Stochastic Programming



Two-stage Stochastic MILP Model

Two-stage Stochastic MILPs have decomposable structure

$$\min \quad z = \sum_{s \in S} p^s \left(c x^s + d^s y^s \right)$$

(λ)

$$\text{s.t.} \quad \boxed{\begin{aligned} & A x^s \geq a && \forall s \in S \\ & B^s y^s + T x^s \geq b^s && \forall s \in S \end{aligned}}$$

$$\boxed{x^s = x^{s'} \qquad \forall \left(s, s'\right) \in S, s < s'}$$

Stage-1 decisions

$$x_j^s \in \mathbb{Z}^+ \qquad \forall s \in S, \forall j \in J'$$

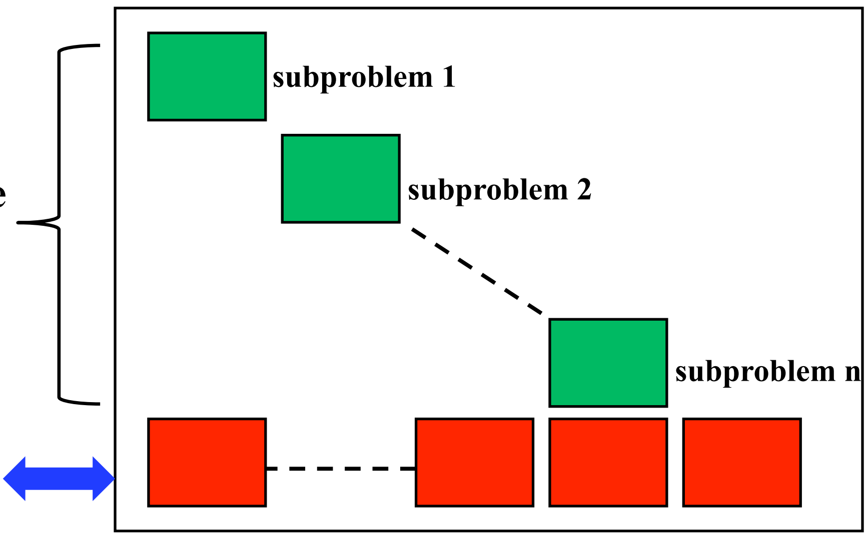
$$x_j^s \in \mathbb{R} \qquad \forall s \in S, \forall j \in J \backslash J'$$

Stage-2 decisions

$$y_k^s \in \mathbb{Z}^+ \qquad \forall s \in S, \forall k \in K'$$

$$y_k^s \in \mathbb{R} \qquad \forall s \in S, \forall k \in K \backslash K'$$

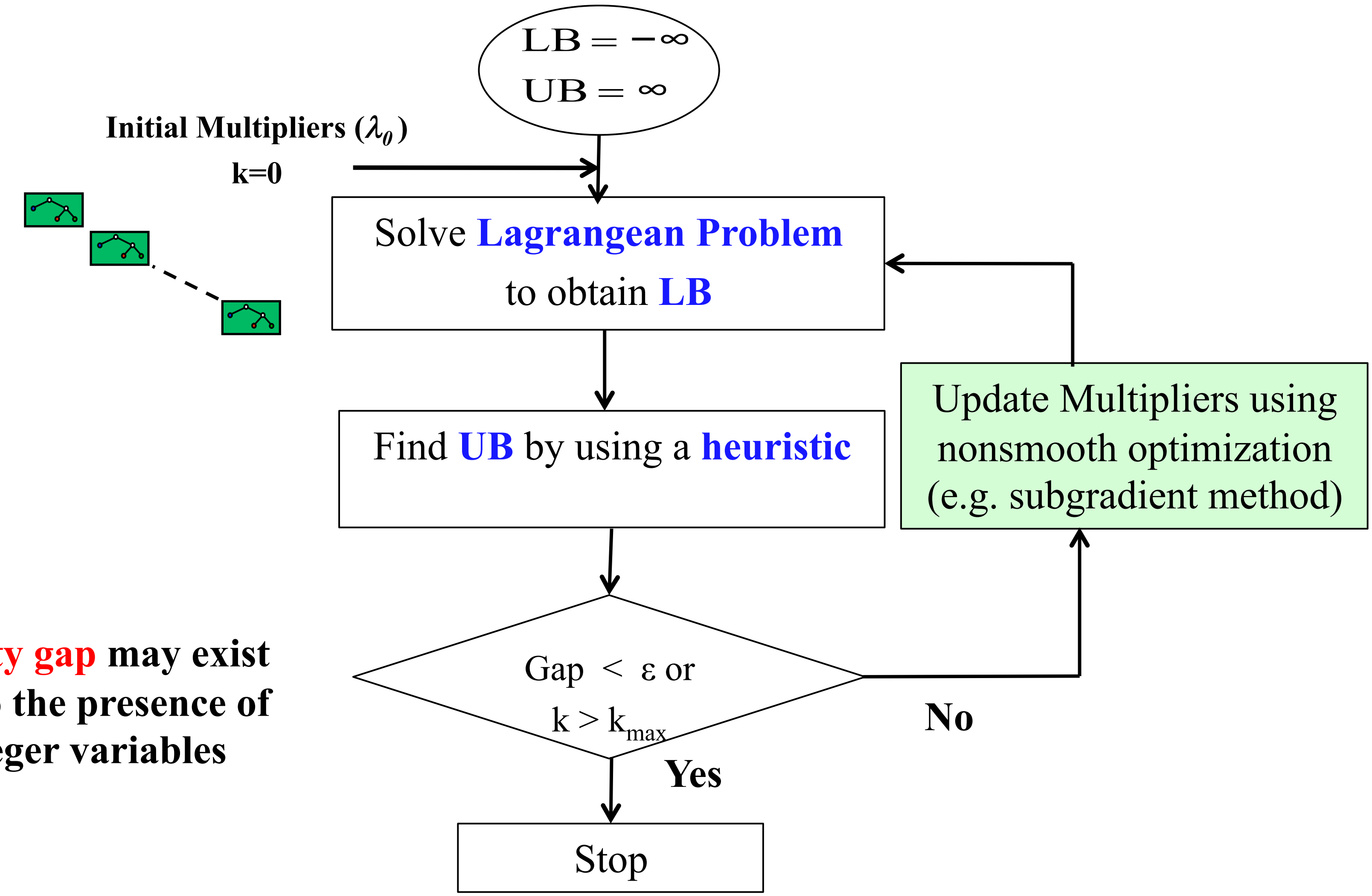
Scenario constraints are in blocks



Non-anticipativity constraints link the scenarios (complicating constraints)

If we dualize the Non-anticipativity constraints in the objective function, problem decomposes into scenarios and we can solve it using Lagrangean Decomposition.

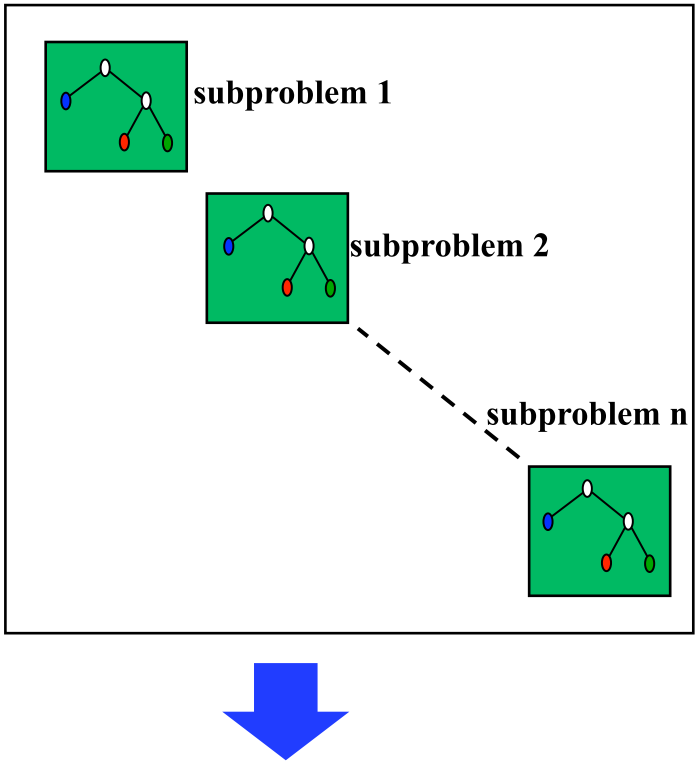
Lagrangean Decomposition Algorithm (Standard)



Duality gap may exist due to the presence of integer variables

Drawbacks of Nonsmooth Optimization

- Using only the optimal solution of each subproblem and discard all relevant information generated during Branch and Bound algorithm
- Slow convergence. Number of iterations required are usually large.
- Need a heuristic procedure to update the step size and upper bound in each iteration
- Some has been improved by Bundle methods, Volume algorithm, etc.



Goal

1. Extract the useful information from branch and bound tree of each subproblem and use it to improve the lower bound efficiently
2. Propose a new algorithm for MILP models with decomposable structure (e.g. 2-stage stochastic) and benchmark the results against subgradient method

Integer Programming (IP) Sensitivity Analysis

IP sensitivity Analysis (Primal Analysis and Dual Analysis) allow us to find valid tight bounds for the objective function value when the objective function coefficients are perturbed, using the information coming from the Branch and Bound solution tree.

Original Problem

$$(P) \quad \min \quad z = cx \qquad \textbf{(1)}$$

$$s.t. \quad Ax \geq a \qquad \textbf{(2)}$$

$$x_j \in \mathbb{Z}^+ \qquad \forall j \in J' \qquad \textbf{(3)}$$

$$x_j \in \mathbb{R} \qquad \forall j \in J \backslash J' \qquad \textbf{(4)}$$

$$L_j \leq x_j \leq U_j \qquad \forall j \in J \qquad \textbf{(5)}$$

Perturbed Problem

$$\left(\hat{P}\right) \quad \min \quad \hat{z} = (c + \Delta c)\, x$$

$$s.t. \quad \textbf{(2) – (5)}$$

Dual Analysis

Primal Analysis

Bounds on the Perturbed Problem using Sensitivity Analysis

$$LB \leq \hat{z} \leq UB$$

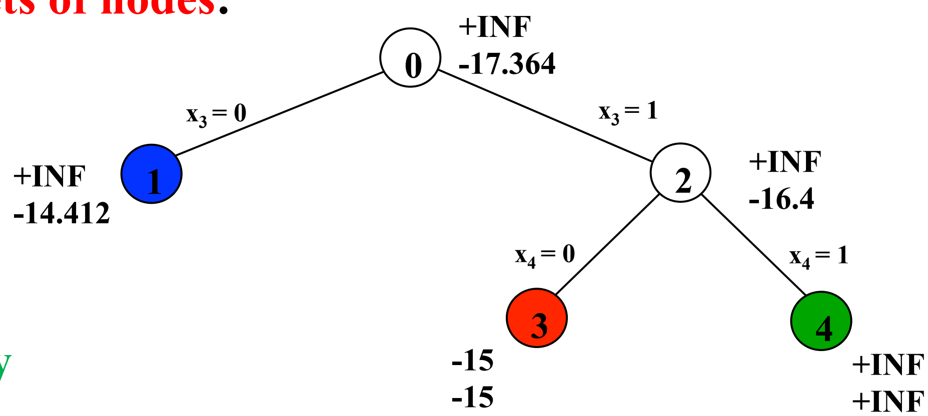
Primal Analysis (PA): Upper Bound

When IP or MIP problem is solved using Branch and Bound method, each leaf node belongs to one of the following three sets of nodes:

N_1: Set of nodes pruned by optimality
(feasible integer solutions)

N_2: Set of nodes pruned by bound
(non-integer feasible solutions)

N_3: Set of nodes pruned by infeasibility
(leaf nodes pruned by infeasibility)



Primal Analysis says that the feasible solutions at any node in N_1 stay feasible (but not necessarily optimal) when the objective function coefficients change to $(c + \Delta c)$. The best feasible solution is the minimum of the available solutions.

$$\begin{aligned} \max \quad & UB \\ s.t. \quad & UB \leq z_n + \sum_{j \in J} v_j^n \, \Delta c_j \qquad \forall n \in N_1 \end{aligned}$$

Dual Analysis (DA): Lower Bound

Dual Analysis involves set of constraints that gives the maximum amount of decrease in the objective when the objective function coefficients are perturbed

The bound $\hat{z} \geq z - \Delta z = LB$ **remains valid if** r_n, s_j^n **satisfy the following set of constraints.**

Parameters from B&B tree

Variables

$$\sum_{j \in J} \Delta c_j \underline{v}_j^n - s_j^n \left(\bar{v}_j^n - \underline{v}_j^n\right) \geq -r_n \qquad n \in N_1 \cup N_2$$

$$r_n = -\sum_{j \in J} q_j^n \bar{v}_j^n + \bar{\lambda}^n a - z_n + \Delta z \qquad \forall n \in N_1$$

$$r_n = -\sum_{j \in J} q_j^n \bar{v}_j^n + \bar{\lambda}^n a - \bar{z}_n + \Delta z \qquad \forall n \in N_2$$

$$s_j^n \geq -\Delta c_j \qquad \forall j \in J, \forall n \in N_1 \cup N_2$$

$$s_j^n \geq -q_j^n \qquad \forall j \in J, \forall n \in N_1 \cup N_2$$

where $q_j^n = \bar{\lambda}^n A - c$

N_1: Set of nodes pruned by optimality

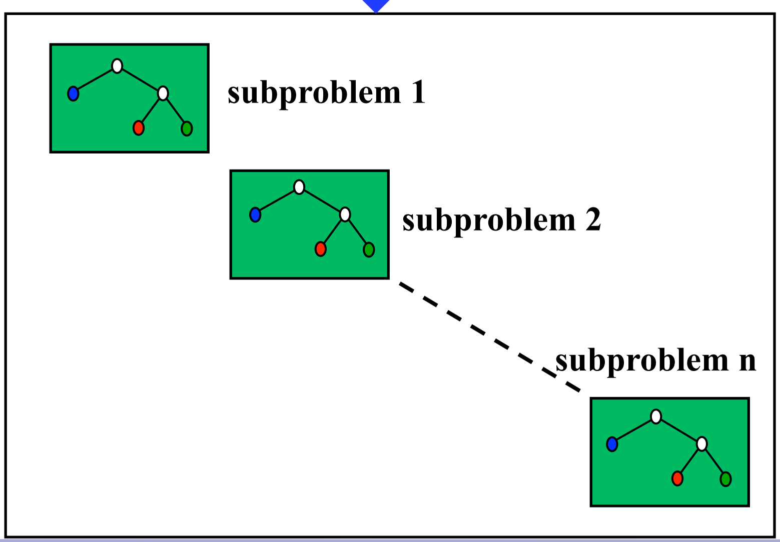
N_2: Set of nodes pruned by bound

(Dawande and Hooker, 1998)

Observation

$$\min \quad z = \sum_{s \in S} p^s \left(\left(c + \boxed{\sum_{s<s'} \lambda^{s,s'} - \sum_{s'<s} \lambda^{s',s}} \right) x^s + d^s y^s \right)$$

$$\begin{array}{lll}
s.t. & Ax^s \geq a & \forall s \in S \\
& B^s y^s + T x^s \geq b^s & \forall s \in S \\
& x_j^s \in \mathbb{Z}^+ & \forall s \in S, \forall j \in J' \\
& x_j^s \in \mathbb{R} & \forall s \in S, \forall j \in J \backslash J' \\
& y_k^s \in \mathbb{Z}^+ & \forall s \in S, \forall k \in K' \\
& y_k^s \in \mathbb{R} & \forall s \in S, \forall k \in K \backslash K'
\end{array}$$



$$\min \quad \hat{z} = (c + \boxed{\Delta c})x$$

$$\begin{array}{lll}
s.t. & Ax \geq a & \\
& x_j \in \mathbb{Z}^+ & \forall j \in J' \\
& x_j \in \mathbb{R} & \forall j \in J \backslash J' \\
& L_j \leq x_j \leq U_j & \forall j \in J
\end{array}$$

IP Sensitivity analysis provide LB and UB on perturbed problem

$$\boxed{\Delta c^s = \sum_{s<s'} \lambda^{s,s'} - \sum_{s'<s} \lambda^{s',s}}$$

The idea is to find such multipliers $\left(\lambda^{s,s'}\right)$ that maximize the sum of upper and lower bounds generated by primal and dual analysis.

Combining Sensitivity Analysis for Multiplier Updating

Proposed Linear Program

- Parameters from B&B tree of each scenario
- Variables
- Adjustable Parameters

Objective

$$\max \; w_1 UB + w_2 LB$$

Primal Analysis (PA)

$$UB = \sum_s UB_s$$

$$UB_s \le \hat{z}_n^{s,k} + \sum_j v_j^{n,s,k} \Delta c_j^s \qquad \forall s \in S, n \in N_1^{s,k}, k \in \{1,2.....k\}$$

Dual Analysis (DA)

$$LB = \sum_{s \in S} (z^s - \Delta z^s)$$

$$\sum_{j \in J} \Delta c_j^s \underline{v}_j^{n,s} - s_j^{n,s} \left(\bar{v}_j^{n,s} - \underline{v}_j^{n,s} \right) \ge -r_n^s \qquad \forall s \in S, n \in N_1^s \cup N_2^s$$

$$r_n^s = -\sum_{j \in J} q_j^{n,s} \bar{v}_j^{n,s} + \bar{\lambda}^{n,s} a - z_n^s + \Delta z^s \qquad \forall s \in S, \forall n \in N_1^s$$

$$r_n^s = -\sum_{j \in J} q_j^{n,s} \bar{v}_j^{n,s} + \bar{\lambda}^{n,s} a - \bar{z}_n^s + \Delta z^s \qquad \forall s \in S, \forall n \in N_2^s$$

$$s_j^{n,s} \ge -\Delta c_j^s \qquad \forall j \in J, \forall s \in S, \forall n \in N_1^s \cup N_2^s$$

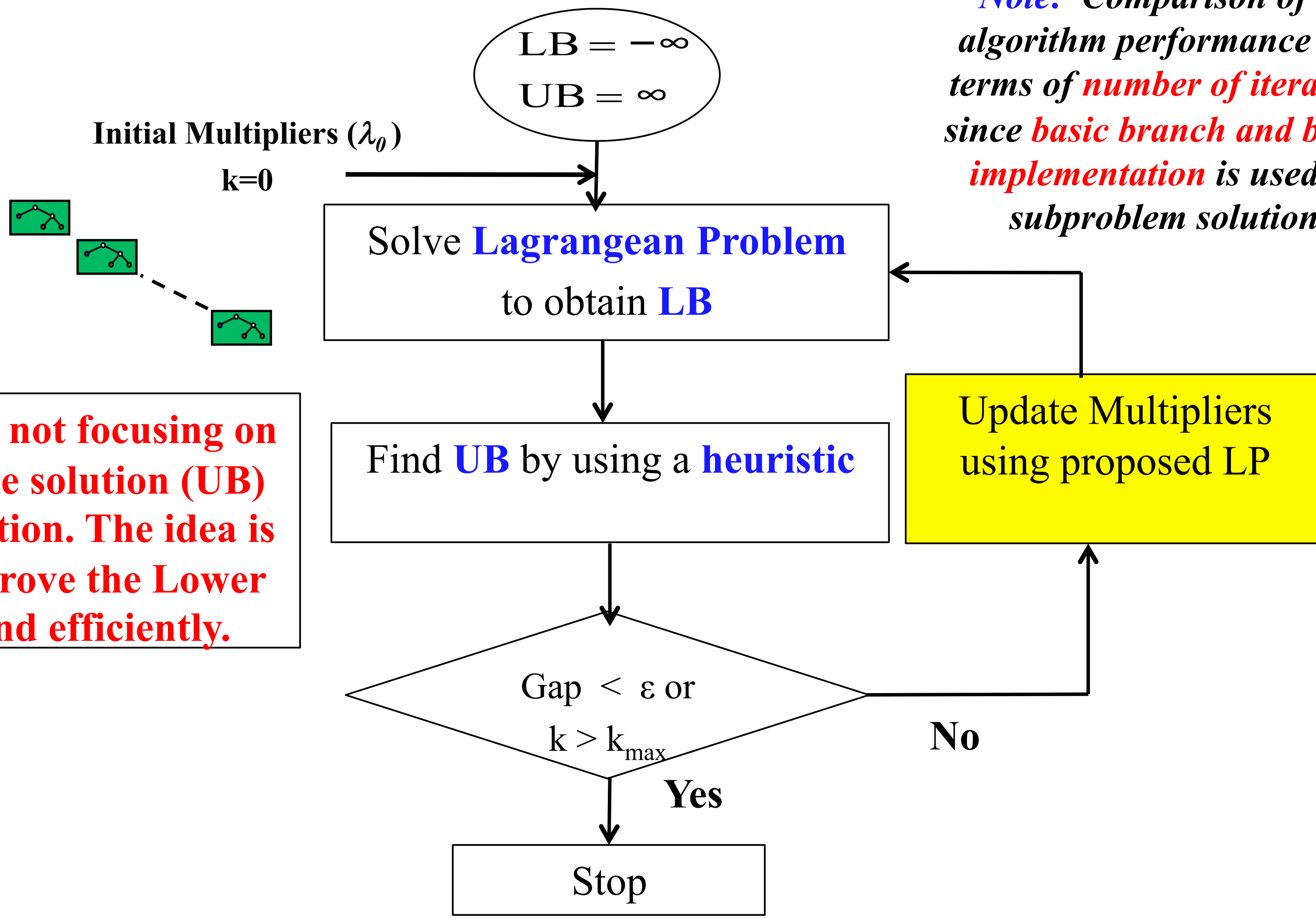
$$s_j^{n,s} \ge -q_j^{n,s} \qquad \forall j \in J, \forall s \in S, \forall n \in N_1^s \cup N_2^s$$

Restriction on Δc_j^s

$$|\Delta c_j^s| \le \gamma \max_{j \in \bar{J}} |c_j^s| \qquad \forall j \in \bar{J}, \forall s \in S$$

Lagrangean Decomposition Algorithm (Proposed)

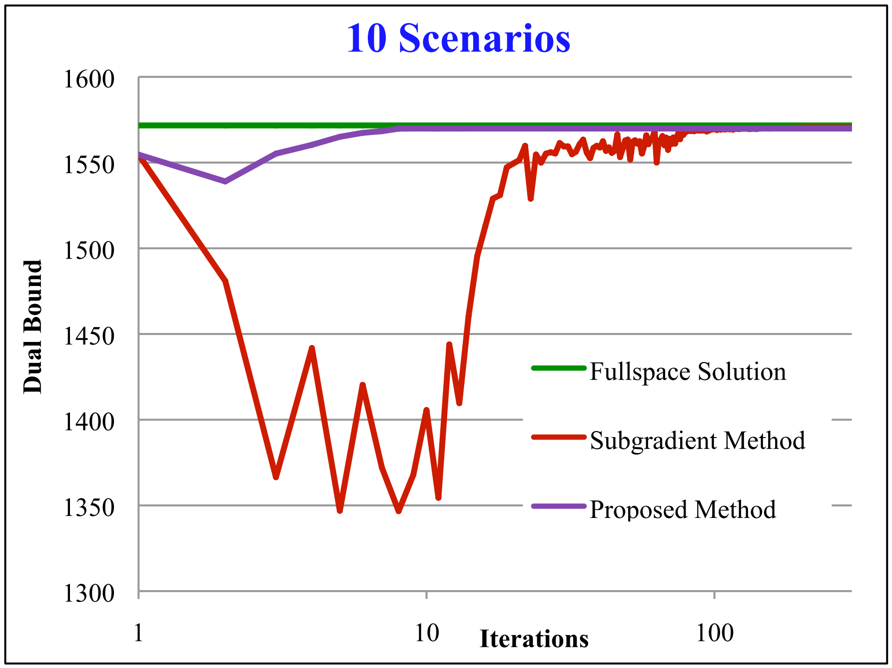
Note: Comparison of the algorithm performance is in terms of number of iterations since basic branch and bound implementation is used for subproblem solutions

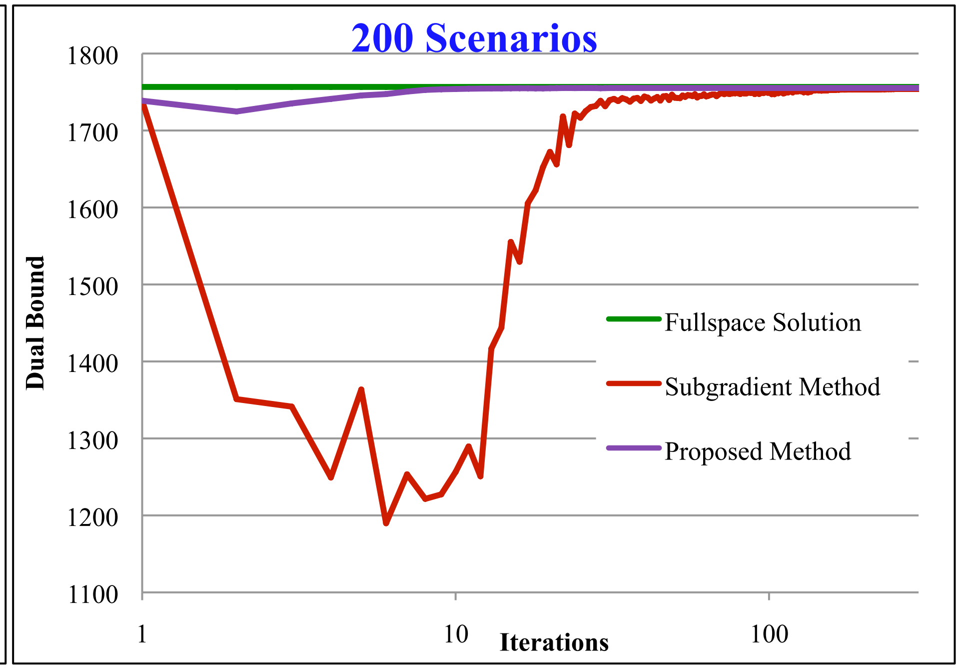


We are not focusing on feasible solution (UB) generation. The idea is to improve the Lower Bound efficiently.

Example (Dynamic Capacity Allocation Problem (DCAP))

	Number of Scenarios	Model Statistics (Deterministic Equivalent)				
		Discrete Variables	Continuous Variables	Constraints	First Stage Variables	Second Stage Variables
DCAP *Ahmed and Garcia (2003)*	10	330	60	840	120	270
	200	6,600	1,200	2,44,800	2,400	5,400





Proposed method outperforms Subgradient method (<10 iterations vs. ~200 iterations)

Conclusions

- A method is proposed for improving the dual bound of decomposable MILP models using sensitivity analysis

- The method outperforms standard subgradient method in terms of number of iterations for stochastic MILPs and significant computational savings can be achieved if optimizing each subproblem takes a long time

- Application to more general class of MILPs (e.g. multistage stochastic) and improvement in the implementation efficiency are part of the future work