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Problem Statement and Main Assumptions

Given
- Plants, Products, Operating Modes and Production Limits
- Daily Electricity Prices (off-peak and peak)
- Customers and their demand/consumption profiles
- Max/Min inventory at production sites
- Alternative sources and product availabilities
- Fixed Planning Horizon (usually 1-2 weeks)

Decisions in each time period \( t \)
- Modes and production rates at each plant
- Inventory level at plants
- How much product to be delivered to each customer through which route

Objective Function
- Minimize total production and distribution cost over planning horizon

Main Assumptions – Distribution Side
- Two time periods per day (peak and off-peak) are considered
- Trucks do not visit more than 4 customers in a single delivery
- Depots, Truck availabilities and capacities, Distances
Sequential vs Simultaneous approach

**Motivation:** Simultaneous optimization reduces the overall total cost but at much higher computational expense.

<table>
<thead>
<tr>
<th>Model Size</th>
<th>Sequential Model</th>
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<tbody>
<tr>
<td></td>
<td>Production</td>
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<tr>
<td>Normalized cost</td>
<td>52.53</td>
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<td><strong>105.03</strong></td>
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<table>
<thead>
<tr>
<th>CPU results</th>
<th>Time</th>
<th>Nodes</th>
<th>Relative gap</th>
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</thead>
<tbody>
<tr>
<td></td>
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<td>2%</td>
</tr>
<tr>
<td></td>
<td>952</td>
<td>-</td>
<td>9.6%</td>
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</table>
### Sequential vs Simultaneous approach

**Motivation:** Simultaneous optimization reduces the overall total cost but at much higher computational expense.

**Tradeoff:** Total cost vs computational effort

<table>
<thead>
<tr>
<th></th>
<th>Sequential Model</th>
<th>Simultaneous Model</th>
<th>Rolling horizon</th>
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<td><strong>CPU results</strong></td>
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<tr>
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<td>9.6%</td>
<td>4.7%</td>
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</tbody>
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![Graph showing tradeoff between total cost and computational effort](image)
Deterministic Production Distribution of Industrial Gases Supply Chains Management

Reduce the computational effort (Large-scale optimization techniques)

Production and distribution coordination (PDC) under demand uncertainty

Deterministic models

In reality we are almost never certain about the parameters
Uncertain parameters are difficult to characterize

Robust Optimization

- A robust program “worst case optimization” focuses on a computationally tractable description of an uncertainty set, against which the solution is “robustified”
- Short term planning (scheduling problems) have limited recursive actions
- Decision variables take values before the actual data “reveals itself”
Robust Optimization Approach

General framework of Linear RO by Bertsimas and Thiele (2006)
- Derived closed form-expressions to include budgets of uncertainty
  - Examples: single station inventory problem, and network case with demand uncertainty.
- Zhang et al. (2014) applied budgets of uncertainty for reserve market of electricity production.

Consider the following problem:
- Subject to data uncertainty

\[
\text{minimize } c'x \\
\text{s.t. } Ax \leq b \\
l \leq x \leq u
\]

Interval data uncertainty
- Each uncertain coefficient
  \[
a_{i,j} \in [\bar{a}_{i,j} - \hat{a}_{i,j}, \bar{a}_{i,j} + \hat{a}_{i,j}]
\]
  - \(\bar{a}_{i,j}\) Nominal value
  - \(\hat{a}_{i,j}\) Deviation from its nominal value
  - Scaled deviation from its nominal value
  \[
w_{i,j} = (a_{i,j} - \bar{a}_{i,j})/\hat{a}_{i,j} \text{ with values in } [-1,1]
\]
- Budget of uncertainty, why budget? The total variation (scaled) cannot exceed some threshold
  \[
  \sum_{(i,j) \in J} |w_{ij}| \leq \Gamma
  \]
  \(\Gamma = 0\) nominal case; \(\Gamma = |J|\) worst case; \(0 < \Gamma < |J|\) provides flexibility
Robust Approach

- The robust problem is then formulated as:

\[ \begin{align*}
U &= \left\{ U \in R^{mxn} \mid a_{ij} \in [\bar{a}_{ij} - \bar{a}_{ij}, \bar{a}_{ij} + \bar{a}_{ij}] \forall i, j, \sum_{j \in J} w_{ij} \leq \Gamma \right\} \\
\mathbf{w}_{i,j} &= (a_{i,j} - \bar{a}_{i,j})/\bar{a}_{i,j} \\
\end{align*} \]

\[ \begin{align*}
\text{minimize} \quad & c'x \\
\text{s. t.} \quad & Ax \leq b \quad \forall A \in U \\
& l \leq x \leq u \\
\end{align*} \]

To ensure robustness requires for \(i^{th}\) constraint to:

\[ \begin{align*}
\text{maximize} \quad & \sum_j (\bar{a}_{i,j} - \mathbf{w}_{i,j} \hat{a}_{i,j}) x_j \\
\text{s. t.} \quad & \sum_j w_{ij} \leq \Gamma \\
& 0 \leq w_{ij} \leq 1 \\
\end{align*} \]

Robust Counterpart (RC) for all \(i\) constraint:

- Inner problem (or auxiliary problem)

\[ \begin{align*}
\text{minimize} \quad & q_i \Gamma + \sum_{i \in J_i} s_{ij} \\
\text{s. t.} \quad & q_i + s_{ij} \geq \hat{a}_{ij} \forall j \in J \\
& q_i \geq 0, s_{ij} \geq 0 \forall j \in J \\
\end{align*} \]

\[ \begin{align*}
\text{minimize} \quad & c'x \\
\text{s. t.} \quad & \sum_j \bar{a}_{i,j} X_J + q_i \Gamma + \sum_{i \in J_i} s_{ij} \leq b_i \quad \forall i \\
& q_i + s_{ij} \geq \hat{a}_{ij} y_i \quad \forall j \in J \\
& l \leq x \leq u \quad -y \leq x \leq y \\
& q \geq 0, s \geq 0, y \geq 0 \\
\end{align*} \]

*If \(\mathbf{b}\) and \(\mathbf{c}\) are uncertain we can rewrite the problem to represent the uncertainty as part of \(A\).
**Uncertain Set**

- PDC model currently considers fixed demand forecast
  - Customers with fixed demands over the time horizon
  - i.e. customer one (u1) has 2 demands: (D1 in time period 3 and D2 in time period 6)
- Let’s consider closed box uncertainty (or a deviation from its nominal value)

Uncertainty set $U$

- RO – Robustifies the solution
  - Optimal solution provides a feasible solution for the entire uncertainty set
  - **Budget of uncertainty**

<table>
<thead>
<tr>
<th>$\bar{D}$ nominal value</th>
<th>$\bar{D}$ deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1$</td>
<td>$D_2$</td>
</tr>
<tr>
<td>$-\bar{D}_1$</td>
<td>$\bar{D}_2$</td>
</tr>
<tr>
<td>$\bar{D}_1$</td>
<td>$-\bar{D}_2$</td>
</tr>
</tbody>
</table>

Worst case

$$\maximize \sum_{t}^{t} \bar{D}_{tr} w_k$$

s.t. $$\sum_{t}^{t} w_{tr} \leq \Gamma \quad \forall k \leq t$$

$$0 \leq w_{tr} \leq 1$$
**Uncertain Set**

- PDC model currently considers fixed demand forecast
  - Customers with fixed demands over the time horizon
  - i.e. customer one (u1) has 2 demands: (D1 in time period 3 and D2 in time period 6)
- Let’s consider closed box uncertainty (or a deviation from its nominal value)

**Uncertainty set U**

- RO – Robustifies the solution
  - Optimal solution provides a feasible solution for the entire uncertainty set
- **Budget of uncertainty**

\[
\Gamma = 2: w_1=1, w_2=1 \\
\Gamma = 1: w_1=1, w_2=0 \\
\Gamma = 1: w_1=0, w_2=1
\]

These budgets of uncertainty rule out some demand deviations, and can be seen as “reasonable worst-case approach”
Robust approach

Uncertainty set: Given a forecasted demand \([D(t)]\), the uncertainty set can be formulated as a deviation of the nominal value for each time period.

\[
[D_t - \hat{D}_t, \bar{D}_t + \hat{D}_t]
\]

- Applying the Budget uncertainty approach to the constraints affected by the uncertainty set:

\[
\sum_{r=1}^{t} \sum_{r'} S_{r,t'} \geq \bar{D}_t + \hat{D}_t w_t, \forall t
\]

Then, we obtain the Dual

- Formulate the auxiliary problem:

\[
\text{maximize} \quad \sum_{t} \hat{D}_{t'} w_{k}
\]

s.t.

\[
\sum_{t'} w_{t'} \leq \Gamma \quad \forall k \leq t
\]

\[
0 \leq w_{t'} \leq 1
\]

- And find the Dual of the auxiliary problem:

\[
\sum_{t \in \mathbb{P}} \sum_{r} S_{r,t'} - \max_{w} \left\{ \sum_{t} \hat{D}_{t',w_t} \right\} \geq \bar{D}_t, \forall t
\]

This auxiliary problem must be solved for all customers.
Robust approach

- Applying the Budget of uncertainty approach to the constraints affected by the uncertainty set:

\[
\sum_{tp=1}^{t} \sum_{r} S_{ur_{tp}} - \max_{w} \left\{ \sum_{t'}^{t} \hat{D}_{ut'}w_{ut'} \right\} \geq \bar{D}_{ut}, \forall u, t
\]

- Formulate the auxiliary problem:

\[
\begin{align*}
& \text{maximize} \quad \sum_{t}^{t} \hat{D}_{ut'} w_{k} \\
& \text{s.t.} \quad \sum_{t'} w_{ut'} \leq \Gamma \quad \forall u, k \leq t \\
& \quad \quad \quad \quad \quad \quad 0 \leq w_{ut'} \leq 1
\end{align*}
\]

Subject to:
- Mode selection
- Start up detection
- Plant inventory
- Production plant adhoc models
- Distribution side contraints

- And find the Dual of the auxiliary problem:

\[
\begin{align*}
& \text{minimize} \quad q_{ut}\Gamma_{u} + \sum_{t'=1}^{t} s_{utt'} \\
& \text{s.t.} \quad q_{ut} + s_{utt'} \geq \hat{D}_{ut} \quad \forall u, t' \leq t \\
& \quad \quad \quad \quad \quad \quad q_{ut} \geq 0, s_{utt'} \geq 0 \quad \forall u, t' \leq t
\end{align*}
\]

\[
\text{minimize Total Cost}
\]

\[
\sum_{tp=1}^{t} \sum_{r} S_{ur_{tp}} - \left\{ q_{ut}\Gamma_{u} + \sum_{t'=1}^{t} s_{utt'} \right\} \geq \bar{D}_{ut}, \forall u, t
\]

\[
\text{s.t.} \quad q_{ut} + s_{utt'} \geq \hat{D}_{ut} \quad \forall u, t' \leq t \\
& \quad \quad \quad \quad \quad \quad q_{ut} \geq 0, s_{utt'} \geq 0 \quad \forall u, t' \leq t
\]
Case study (PDC – under demand uncertainty)

- 4 Plants / Depots
- 2 products (Lin, Lox)
- 2-3 production modes for each plant
- 15 alternative sources
- 105 customer
- 14 time periods (peak and off peak)
- 46 trucks (25 for LIN, 12 for LOX)
- Min/Max inventory, distances, electricity prices, truck deliveries, etc.
Demand uncertainty – robust optimization results

- Nominal Demand value:

![Graph showing Demand over time with values for u1 and u10]

Maximum truck capacity (665; 700)

Strategy

- **Worst case** (deviation = 20% of nominal value,)
  - Nominal Value + 20%
  - $\Gamma = \text{max # of orders}$

Budget uncertainty (deviation = 20% of nominal value)

- Nominal Value + 20%
- $\Gamma = 1$
Demand uncertainty – robust optimization results

- **Nominal Demand value:**

  - Nominal value: 575
  - Nominal + Deviation: 575 * 1.2 = 690

- **Worst case** (deviation = 20% of nominal value, $\Gamma_{max}$ # of orders)

  - Distribution (customer through route $r$)

  - Nominal value: 575
  - Nominal + Deviation: 575 * 1.2 = 690

**Model Predictions**

- Distribution: route $r_{231}$ at $t_9$: 665
- Distribution: route $r_{231}$ at $t_{10}$: 25
- Total product distributed: 690

Customer 1
### Demand uncertainty – robust optimization results

#### Nominal Demand value:

- Nominal value: 665
- Nominal + Deviation: 665 * 1.2 = 798

#### Worst case (deviation = 20% of nominal value, $\Gamma = \text{max } \# \text{ of orders}$)

- Distribution (customer through route r)

- Model Predictions
  - Distribution: route r255 at t1: 665
  - Distribution: route r281 at t2: 133
  - Total product distributed: 798
Demand uncertainty – robust optimization results

- **Nominal Demand value:**

  - **Budget uncertainty approach** (deviation = 20% of nominal value, $\Gamma=1$)

  - **Customer 10**

  - **NOMINAL VALUE**
    - Nominal value: 665
    - Nominal + Deviation: 665*1.2: 798

  - **Model Predictions**
    - Distribution: route $r_{254}$ at $t_1$: 665
    - Distribution: route $r_{300}$ at $t_2$: 133
    - Total product distributed: 798

- **Demand**

- **Distribution (customer through route r)**

  - (max truck cap)
Results analysis

Customer 1

• Worst case optimization ($\Gamma = 1$).

  Distribution: route r231 at time t9: 665
  Distribution: route r231 at time t10: 25
  Total product distributed: 690

• Budget uncertainty ($\Gamma = 1$).

  Distribution: route r222 at time t9: 690
  Total product distributed: 690

Depot 3

k20 Load 25 at plant 3

Route r222 from plant 3, k21: 25
Route r222 from alt. source: 665
Total Product distributed: 690

k21 Load 665 at alternative source 8
Results analysis

Total
Distributed from plant 1 and product 1 in time t5:
3,039 units

To total Distributed from plant 1 and product 1 in time t5:
1,401 units

Customer 2

Worst case
u2 = 204
u50 = 328
u199 = 133
k3 = 665

Loaded at plant 1 in t5

Budget uncertainty
u2 = 204
u50 = 260
u205 = 133
k3 = 603

Loaded at plant 1 in t5
Summary

- 4 Plants / Depots
- 2 products (Lin, Lox)
- 2-3 production modes for each plant
- 15 alternative sources
- 105 customer demands
- 14 time periods (peak and off peak)
- 46 trucks (25 for LIN, 12 for LOX)
- Min/Max inventory, distances, electricity prices, truck deliveries, etc.

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<tr>
<th>Model Size</th>
<th>Original</th>
<th>Worst case</th>
<th>Budget</th>
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<td>Total cost</td>
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<tr>
<td></td>
<td>15,330</td>
<td>12%</td>
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There is a reasonable tradeoff between robustness and performance.
Proposed framework provides optimal production-distribution coordination under uncertainty

Robust optimization framework

- Powerful decision making tool
  - Scheduling problems (recursive actions are unrealistic)
    - Complexity: large scale problems cannot include scenario realization of the uncertain parameters (tractability)
    - Product availability limitations (purchase product from alternative sources)