Tutorial on Lagrangean Decomposition: Theory and Applications

Ignacio Grossmann and Bora Tarhan
Center for Advanced Process Decision-making
Department of Chemical Engineering
Carnegie Mellon University
Pittsburgh, PA 15213
Decomposable MILP Problems

### Complicating Constraints

\[
\begin{align*}
\text{max } & \quad c^T x \\
\text{st } & \quad Ax = b \\
& \quad D_i x_i = d_i \quad i = 1,..n \\
& \quad x \in X = \{x \mid x_i, i = 1,..n \mid x_i \geq 0\}
\end{align*}
\]

### Complicating Variables

\[
\begin{align*}
\text{max } & \quad a^T y + \sum_{i=1}^{n} c_i^T x_i \\
\text{st } & \quad Ay + D_i x_i = d_i \quad i = 1,..n \\
& \quad y \geq 0, \quad x_i \geq 0, \quad i = 1,..n
\end{align*}
\]

**Benders decomposition**

*Note: can reformulate by defining*

\[
y_i = y_{i+1} \quad \text{Complicating constraints}
\]

*and apply Lagrangean decomposition*
Joseph Louis Lagrange
January 25, 1736 – April 10, 1813

Born in Turin, Italy: Italian parents (French ancestors father side)
Born: Giuseppe Lodovico Lagrangia

1766: Lagrange succeeded Euler as director of mathematics at the Prussian Academy of Sciences in Berlin
1794: Became the first professor of analysis at the opening of École Polytechnique
1808: Napoleon named him to the Legion of Honour and made him a Count of the Empire in 1808
Lagrangean or Lagrangian Decomposition?

Google hits:
– “Lagrangian” returned 2,190,000 hits.
– “Lagrangean” returned 104,000 hits.

Google hits:
– “Lagrangian decomposition” returned 671,00 hits.
– “Lagrangean decomposition” returned 1,580,000 hits.

Google hits:
– “Lagrangian relaxation” returned 133,000 hits.
– “Lagrangean relaxation” returned 275,000 hits.

We will spell it as “Langrangean”
Major references on Lagrangean Relaxation/Lagrangian Decomposition

Mathematical Programming Study 2 (1974) 82

Interfaces (1985) 15, 10.

<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Title</th>
<th>Journal</th>
<th>Year</th>
<th>Pages</th>
</tr>
</thead>
</table>
MILP optimization problems can often be modeled as problems with complicating constraints.

The complicating constraints are added to the objective function (i.e. dualized) with a penalty term (Lagrangean multiplier) proportional to the amount of violation of the dualized constraints.

The Lagrangean problem is easier to solve (eg. can be decomposed) than the original problem and provides an upper bound to a maximization problem.
Lagrangean Relaxation

\[ Z = \max cx \]
\[ Ax \leq b \]
\[ Dx \leq e \]
\[ x \in \mathbb{Z}^n_+ \]

Assume that \( Ax \leq b \) is complicating constraint

\[ Z_{LR}(u) = \max cx + u(b - Ax) \]
\[ Dx \leq e \]
\[ x \in \mathbb{Z}^n_+ \]

where \( u \geq 0 \) Lagrange multipliers
Lagrangean Relaxation

\[ Z = \max cx \quad Z_{LR}(u) = \max cx + u(b - Ax) \]

Complicating Constraint

\[ Ax \leq b \quad Dx \leq e \]

\[ Dx \leq e \quad x \in \mathbb{Z}^n \quad x \in \mathbb{Z}_+^n \]

where \( u \geq 0 \)

This is a relaxation of original problem because:

i) removing the constraint \( Ax \leq b \) relaxes the original feasible space,

ii) \( Z_{LR}(u) \geq Z \) always holds as in the original space since \( (b - Ax) \geq 0 \)

and Lagrange multiplier is always \( u \geq 0 \).

\[ \text{Lagrangean Relaxation Yields Upper Bound} \quad \Rightarrow \quad Z_{LR}(u) \geq Z \]
Lagrangian Relaxation

Original problem:
\[ Z = \max \ cx \]
\[ Ax \leq b \]
\[ Dx \leq e \]
\[ x \in Z^n_+ \]

Relaxed problem:
\[ Z_{LR}(u) = \max \ cx + u(b - Ax) \]
\[ Dx \leq e \]
\[ x \in Z^n_+ \]

Lagrangean dual:
\[ Z_D = \min Z_{LR}(u) \]
\[ u \geq 0 \]
Relaxed problem:

\[ Z_{LR}(u) = \max \quad cx + u(b - Ax) \]
\[ Dx \leq e \]
\[ x \in \mathbb{Z}_+^n \]

Lagrangean dual:

\[ Z_D = \min Z_{LR}(u) \]
\[ u \geq 0 \]

Combine Relaxed and Lagrangean Dual Problems:

\[ Z_D = \min \left\{ \max_{u \geq 0} \max_{x \geq 0} \quad cx + u(b - Ax) \right\} \]
\[ Dx \leq e \]
\[ x \in \mathbb{Z}_+^n \]
Graphical Interpretation

\[ Z_D = \min_{u \geq 0} \left\{ \max_{x \geq 0} cx + u(b - Ax) \right\} \quad Z_D' = \max cx \]
\[ Ax \leq b \quad x \in \text{Conv}(Dx \leq e, x \in Z^+_n) \]
\[ x \geq 0 \]

Optimization of Lagrange multipliers (dual) can be interpreted as optimizing the primal objective function on the intersection of the convex hull of non-complicating constraints set \( \{x \mid Dx \leq e, x \in Z^n_+\} \) and the LP relaxation of the relaxed constraints set \( \{x \mid Ax \leq b, x \in Z^n_+\} \).
Graphical Interpretation

\[ Z_D' = \max cx \]
\[ Ax \leq b \]
\[ x \in \text{Conv}(Dx \leq e, x \in Z^n) \]
\[ x \geq 0 \]
Lagrangean relaxation yields a bound at least as tight as LP relaxation

\[ Z(P) \leq Z_D \leq Z_{LR}(u) \leq Z_{LP} \]
Lagrangian Decomposition (Guignard & Kim, 1987)

- Lagrangian Decomposition is a special case of Lagrangian Relaxation.
- Define variables for each set of constraint, add constraints equating different variables (new complicating constraints) to the objective function with some penalty terms.

\[
Z = \max \ ax \quad Z' = \max \ ax \quad Z_{LD} (v) = \max \ ax + v(y - x)
\]

\[
A x \leq b \\
D x \leq e \\
x \in Z^n_+ \\
y \in Z^n_+
\]

\[
A x \leq b \\
D y \leq e \\
x = y \\
x \in Z^n_+ \\
y \in Z^n_+
\]

Dualize \( x = y \)
Lagrangian Decomposition

\[ Z_{LD}(v) = \max \quad cx + v(y - x) \]
\[ Ax \leq b \]
\[ Dy \leq e \]
\[ x \in Z^n_+ \]
\[ y \in Z^n_+ \]

Subproblem 1

\[ Z_{LD1}(v) = \max \quad (c - v)x \]
\[ Ax \leq b \]
\[ x \in Z^n_+ \]

Subproblem 2

\[ Z_{LD2}(v) = \max \quad vy \]
\[ Dy \leq e \]
\[ y \in Z^n_+ \]

\[ Z_{LD} = \min_{v \geq 0} \quad (Z_{LD1}(v) + Z_{LD2}(v)) \]

Lagrangian dual
Lagrangean decomposition is different from other possible relaxations because every constraint in the original problem appears in one of the subproblems.

Graphically: The optimization of Lagrangean multipliers can be interpreted as optimizing the primal objective function on the intersection of the convex hulls of constraint sets.
Graphical Interpretation?

Subproblem 1

\[ \text{Conv}\{ x | Dx \leq e, x \in \mathbb{Z}_+^n \} \]

\[ \{ x | Dx \leq e \} \]

Subproblem 2

\[ \text{Conv}\{ x | Ax \leq b, x \in \mathbb{Z}_+^n \} \]

\[ \{ x | Ax \leq b \} \]

Note: \( Z_{LR}, Z_{LD} \) refer to dual solutions
The bound predicted by “Lagrangean decomposition” is at least as tight as the one provided by “Lagrangean relaxation” (Guignard and Kim, 1987).

For a maximization problem

\[ Z(P) \leq Z_{LD} \leq Z_{LR} \leq Z_{LP} \]

**Solution of Dual Problem**

- Piecewise linear
- Non-differentiable

![Graph showing piecewise linear behavior with a minimum point](image)
How to iterate on multipliers u?

Assuming $Dx \leq d$ is a bounded polyhedron (polytope) with extreme points $x^k, k = 1, 2\ldots K$, then

$$\max \{ cx + u(b - Ax) \mid Dx \leq d, x \in X \} = \max_{k=1, \ldots, K} \{ cx^k + u(b - Ax^k) \}$$

Dual problem

$$\min_{u \geq 0} \max_{k=1, \ldots, K} \{ cx^k + u(b - Ax^k) \} = \min_{u \geq 0} \{ \eta \mid \eta \geq cx^k + u(b - Ax^k), k = 1, \ldots, K \}$$

Cutting plane approach

$$\min \eta$$

s.t. \( \eta \geq cx^k + u(b - Ax^k), k = 1, \ldots, K_n \)

\( u \geq 0, \eta \in R^1 \)

Note: $x^k$ generated from $\max \{ cx + u^k(b-Ax) \}$ subproblems
Subgradient Optimization Approach

Subgradient \[ s^k = (b - Ax^k) \]

Steepest ascent search \[ u^{k+1} = u^k + \mu s^k \]

**Update formula for multipliers** *(Fisher, 1985)*

\[ u^{k+1} = u^k + \alpha_k (Z^{LB}_k - Z^{LD}_k)(b - Ax^k) / \|b - Ax^k\|^2 \]

where \( \alpha_k \in [0, 2] \)

Note: Can also use bundle methods for nondifferentiable optimization

*Lemarechal, Nemirovski, Nesterov (1995)*
### Solution of Langrangean Decomposition

1. **Iterative search in multipliers of dual**
   - Select $MaxI$, $\varepsilon$, $a^k$
   - Set $UB = +\infty$, $LB = -\infty$
   - Solve $(RP')$ to find $v^0$

   
   - For $k = 1..K$
   - **Solve (P1) and (P2): Obtain $Z_{LD}$**
   - **Solve (P) with fixed binaries or use heuristics: Obtain $Z^L_B$**

   - $|Z_{LD} - Z^L_B| < \varepsilon$? or $k = MaxI$?
   - YES
     - Return $Z^L_B$ & Current Solution
   - NO
     - $k = k+1$
     - Update $u^k$

2. **Perform branch and bound search**
   - where LP relaxation is replaced by Lagrangean relaxation/decomposition to
     a) Obtain tighter bound
     b) Decompose MILP

   Typically in Stochastic Programming
   - Caroe and Schultz (1999)
   - Goel and Grossmann (2006)
   - Tarhan and Grossmann (2008)

### Remarks
1. Methods can be extended to NLP, MINLP
2. Size of dual gap depends greatly on how problems are decomposed
3. From experience gap often decreases with problem size.

### Notes
- Heuristic due to dual gap
- Obtaining Lower Bound might be tricky
Design and Planning of Offshore Oil Infrastructures

Van den Heever (2001)

Design decisions:
WP and pipeline selections
WP type (allows connection or not)
Capacities of PP, WPs and pipelines

Planning decisions:
Investment timing for WPs, pipelines
Production profile in each time period

Objective: Maximize NPV

Complicating factor: Complex economic objectives
Model elements (Multiperiod MINLP)

Objective: Maximize Net Present Value

Reservoir and surface constraints for all $t$, e.g.
- Oil production vs. deliverability
- Deliverability vs. reservoir and surface pressure (nonlinear)
- Reservoir pressure vs. cumulative production (nonlinear)
- Mass balances
- Pressure balances

Logical conditions for all $t$, e.g.
- Each WP only invested in once
- Each WP connected to another WP or PP

Economic calculations for all $t$, e.g.
- Sales revenue
- Capital cost
- Operating cost
- Taxes
- Tariffs
- Royalties


Complex economics
Solution method: Lagrangean Decomposition

Motivation: Complex economics not linked between WPs!
Linking variables - Flows, Pressures

Equations to dualize

\[ Q_{wp,3,t} = Q_{P,wp,3,t} \]
\[ P_{out,wp,t} = P_{out,P,wp,t} \]
(wp = 1,2)

New objective = Old objective + \[ \sum_{wp,t} \left[ \lambda_{wp,t}^Q Q_{P,wp,3,t} - Q_{wp,3,t} \right] + \lambda_{wp,t}^P (P_{out,P,wp,t} - P_{out,wp,t}) \]

Lagrangean multipliers

Model decomposes - one model each WP
Example: 16 WPs, 15 time periods

Full space yields no solution in > 5 days
### Example: Results

<table>
<thead>
<tr>
<th>Constraints</th>
<th>12696</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuous vars.</td>
<td>8552</td>
</tr>
<tr>
<td>Binary vars.</td>
<td>759</td>
</tr>
<tr>
<td>Solution time (CPU h.)</td>
<td>12.3</td>
</tr>
<tr>
<td>NPV ($ mil.)</td>
<td>1219</td>
</tr>
</tbody>
</table>

$95$ million (8.5 %) increase in NPV compared to simple economics!

### Table 9. Sequence of Bounds from the Lagrangean Decomposition Algorithm

<table>
<thead>
<tr>
<th>iteration</th>
<th>Lagrangean soln ($ mil.)</th>
<th>postulated LB ($ mil.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>NLP relaxation</td>
<td>2271.0</td>
<td>*</td>
</tr>
<tr>
<td>2</td>
<td>1611.8</td>
<td>*</td>
</tr>
<tr>
<td>4</td>
<td>1363.7</td>
<td>1164.4</td>
</tr>
<tr>
<td>6</td>
<td>1295.7</td>
<td>*</td>
</tr>
<tr>
<td>8</td>
<td>1265.2</td>
<td>1174.4</td>
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<tr>
<td>10</td>
<td>1254.8</td>
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<td>12</td>
<td>1248.6</td>
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<td>1217.2</td>
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<td>1107.5</td>
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<td>18</td>
<td>1243.3</td>
<td>1218.7</td>
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<tr>
<td>20</td>
<td>1242.8</td>
<td>1117.7</td>
</tr>
</tbody>
</table>

1.9% gap after 20 iterations
Example: Solution

Years 1 to 5

F

B

A

C

Shore

Year 4
Example 2: Solution

Years 6 to 10

Shore
Example: Solution

Years 11 to 15

Shore
NEW PRODUCT DEVELOPMENT

*MOTIVATION:*  
- Increased importance of New Product Development  
- Testing of Pharmaceutical and Agrochemical products  
- Design and modification batch plants

**GOAL:** Develop a new integration model and solution method for Simultaneous Optimization of Planning and Design Manufacturing

**DECISIONS:**  
1. Which products to test and how to test them  
2. In what plants to invest and when  
3. Production profiles
**TRADE-OFFS**

### Testing in New Product Development

**Objective:** Find the schedule that maximizes the NPV  
**Difficulty:** Probability failing tests

#### Three tasks to schedule

<table>
<thead>
<tr>
<th>Task</th>
<th>Cost $c_i$</th>
<th>Probability $p_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Groundwater Studies</td>
<td>$200,000$</td>
<td>$0.7$</td>
</tr>
<tr>
<td>Acute Toxicology</td>
<td>$700,000$</td>
<td>$0.4$</td>
</tr>
<tr>
<td>Formulation Chemistry</td>
<td>$100,000$</td>
<td>$0.9$</td>
</tr>
</tbody>
</table>

Expected Value: $1,000,000$

#### TRADE-OFF: Time to market vs cost

### Design of Process Network

**Objective:** Find the optimal investment strategy  
**Difficulty:** Risk associated with investment

#### DESIGN ALTERNATIVES:

- **Retrofit existing plants**
- **Expand existing plants**
- **Build new plants**

#### TRADE-OFF: Early/large investment vs risk
PROPOSED MILP MODEL

Max NPV = Income - C_{TEST} - C_{INVEST} - C_{PURCH} - C_{OPER}

\begin{align*}
C_{INVEST} &= \sum_p \sum_i \alpha N_{pt} y_{NP_{pt}} + \sum_i \sum_t (\alpha E_{it} y_{E_{it}} + \beta_{it} Q_{E_{it}}) \\
Income &= \sum_m P^m \{\sum_j \sum_t \gamma_{jit} S_{jitm}\}
\end{align*}

\begin{align*}
s_k + d_k z_j - s_{k'} - U (w_j - y_{kk'}) &\leq 0 \\
y_{kk'} &= z_p y_{k'k} = 0 \\
s_k + d_k w_j &\leq T_j \\
y_{kk'} + y_{k'k} &\leq w_j \\
\sum_{q \in (QT(k) \cap QC(r))} \hat{x}_{kq} &= N_{kr} (w_j - x_k) \\
\hat{x}_{kq} + \hat{y}_{kk'} - \hat{y}_{k'k} &\leq 1
\end{align*}

\begin{align*}
y_{E_{it}Q_{E_{it}}^L} &\leq Q_{E_{it}} \leq y_{E_{it}Q_{E_{it}}^U} \\
Q_{it} &= Q_{it-1} + Q_{E_{it}} \\
y_{E_{it}} &\leq \sum_{t' \leq t} y_{NP_{pt}'} \\
\sum_i PP_{jitm} + \sum_{i \in O(j)} W_{ijtm} &= \sum_t SS_{jitm} + \sum_{i \in I(j)} W_{ijtm} \\
W_{ijtm} &= \sum_{s \in PS(i)} \mu_{js} \rho_{is} \theta_{istm} \\
\sum_{s \in PS(i)} \theta_{istm} &\leq Q_{it} H_{P_{it}} \\
\sum_j z_{jt} &= w_j \\
T_j &= \sum_t T_j' \\
HT_{t-1} z_{jt} &\leq T_j' \leq HT_t z_{jt} \\
S_{jitm} &\leq d_{jit} U f_{mj} H_t \sum_{t' \leq t} z_{jt'} \\
S_{jitm} &\leq (HT_t - T_j') d_{jit} U f_{mj}
\end{align*}

\begin{align*}
\forall j \in JP, \forall k, k' \in K(j) \\
\forall j \in JP, \forall k, k' \in K(j), \forall (k, k') \in A \\
\forall j \in JP, \forall k \in K(j) \\
\forall j \in JP, \forall k, k' \in K(j) \mid k < k' \\
\forall j \in JP, \forall k \in K(j), \forall r \in R \\
\forall q, \forall k \in K(q), \forall k' \in (K(q) \setminus KK(k)) \mid k < k'
\end{align*}
SOLUTION METHOD

- Constraint matrix special structure that can be exploited

Use Lagrangean Decomposition to get two independent problems
  - (Fisher, 1985; Guignard & Kim, 1987)

- Solve two decoupled problems separately (computationally cheaper)
- Combine independent solutions to obtain solution of integrated problem

\[ x^1, x^2, x^3 : \text{Vectors of continuous variables} \]
\[ y^1, y^2 : \text{Vectors of discrete variables} \]
Formulation Decomposed MILP

(P) \[ \text{Max NPV} = c^1x^1 + c^2x^2 + c^3x^3 + dy^2 \]
\[ \text{s.t. } A^1x^1 + A^{31}x^3 + B^1y^1 \leq b^1 \quad (I) \]
\[ A^2x^2 + A^{32}x^3 + B^2y^2 \leq b^2 \quad (II) \]

(P') \[ \text{Max NPV} = c^1x^1 + c^2x^2 + c^3x^3 + dy^2 - \lambda(x^{32}-x^{31}) \]
\[ \text{s.t. } A^1x^1 + A^{31}x^3 + B^1y^1 \leq b^1 \quad (I) \]
\[ A^2x^2 + A^{32}x^3 + B^2y^2 \leq b^2 \quad (II) \]
\[ x^{31} = x^{32} \quad (III) \]

Scheduling Subproblem

(P1) \[ \text{Max NPV1} = c^1x^1 + \lambda x^{31} \]
\[ \text{s.t. } A^1x^1 + A^{31}x^{31} + B^1y^1 \leq b^1 \quad (I) \]

Design/Planning Subproblem

(P2) \[ \text{Max NPV2} = c^2x^2 + c^3x^{32} + dy^2 - \lambda x^{32} \]
\[ \text{s.t. } A^2x^2 + A^{32}x^{32} + B^2y^2 \leq b^2 \quad (II) \]

\[ \lambda: \text{Lagrange Multipliers} \]
EXAMPLE: Process Network

Existing Proteins: A, B, D, E
Potentially New Proteins: C, F

<table>
<thead>
<tr>
<th>Process Design Data</th>
<th>Process</th>
<th>Conversion</th>
<th>Capacity (kg/month)</th>
<th>Fixed ($10^3)</th>
<th>Variable ($10^3$-month/kg)</th>
<th>Operating ($10^3$/kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>0.77</td>
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<td>4,500</td>
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<table>
<thead>
<tr>
<th>Prices of Chemicals ($10^3$/kg)</th>
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<th>Price</th>
<th>Product</th>
<th>Price</th>
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<table>
<thead>
<tr>
<th>Demand forecasts (kg/month)</th>
<th>Product</th>
<th>1st yr</th>
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<th>3rd yr</th>
<th>4th yr</th>
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</table>
EXAMPLE: Testing Tasks

Product C

<table>
<thead>
<tr>
<th>Test</th>
<th>Cost ($10^3$)</th>
<th>Cost of outsourcing ($10^4$)</th>
<th>Duration (months)</th>
<th>Prob/ility of success</th>
<th>Resource Req/ment</th>
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<td>80</td>
<td>160</td>
<td>5</td>
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<td>620</td>
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<td>9</td>
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Product F

<table>
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<tr>
<th>Test</th>
<th>Cost ($10^3$)</th>
<th>Cost of outsourcing ($10^4$)</th>
<th>Duration (months)</th>
<th>Prob/ility of success</th>
<th>Resource Req/ment</th>
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<tr>
<td>11</td>
<td>160</td>
<td>320</td>
<td>3</td>
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<td>12</td>
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<td>2260</td>
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<td>0.91</td>
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<td>1060</td>
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<td>230</td>
<td>460</td>
<td>3</td>
<td>1</td>
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</table>
Solution Scheduling Tests & Design Batch Plant

HEURISTIC: Min Completion Time ⇒ $T_c = 10$, $T_f = 17$, $NPV = \$7.62\ billion$

PROPOSED MILP MODEL: $T_c = 10$, $T_f = 22$, $NPV = \$8.12\ billion\ (\pm 7\%)$

TESTING: Gantt Charts for Resources

DESIGN: Expansions of Processes

HEURISTIC:

\[ T_c = 10, \quad T_f = 17, \quad NPV = \$7.62\ billion \]

PROPOSED MILP MODEL:

\[ T_c = 10, \quad T_f = 22, \quad NPV = \$8.12\ billion\ (\pm 7\%) \]
COMPUTATIONAL RESULTS

<table>
<thead>
<tr>
<th></th>
<th>Example 1</th>
<th>Example 2</th>
<th>Example 3</th>
</tr>
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<tbody>
<tr>
<td><strong>Binary Variables</strong></td>
<td>236</td>
<td>354</td>
<td>612</td>
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<tr>
<td><strong>Continuous Variables</strong></td>
<td>9,372</td>
<td>7,456</td>
<td>32,184</td>
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<td><strong>Constraints</strong></td>
<td>8,255</td>
<td>6,825</td>
<td>30,903</td>
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<tr>
<td><strong>Full Space Method</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Optimal Solution</td>
<td>9,518</td>
<td>94,986</td>
<td>135,024</td>
</tr>
<tr>
<td>CPU Time (sec)</td>
<td>8.9</td>
<td>57.2</td>
<td>836.6</td>
</tr>
<tr>
<td><strong>Decomposition Heuristic</strong></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Best Solution</td>
<td>9,518</td>
<td>94,986</td>
<td>134,834</td>
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<tr>
<td>Major iterations</td>
<td>5</td>
<td>3</td>
<td>6</td>
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<tr>
<td>CPU Time (sec)</td>
<td>20.5</td>
<td>37.6</td>
<td>252.4</td>
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<tr>
<td>Relative Duality Gap</td>
<td>3.96%</td>
<td>0.10%</td>
<td>0.77%</td>
</tr>
</tbody>
</table>

![Graph showing duality gap over iterations for Ex 1, Ex 2, and Ex 3]
Objective: Develop model and effective solution strategy for large-scale multiperiod planning with *Nonlinear Process Models*
Multisite Distribution Model

• Develop Multisite Model to determine:
  1) What products to manufacture in each site
  2) What sites will supply the products for each market
  3) Production and inventory plan for each site

  ➢ *Objective: Maximize Net Present Value*

• Challenges/Optimization Bottlenecks: Large-Scale NLP
  – Interconnections between time periods & sites/markets
  ➢ Apply *Lagrangian Decomposition Method*
Spatial Decomposition

\[
\text{max } \text{PROFIT} = S\text{Cost}_{S}^{PR,M} \cdot \text{SALES}_{S}^{PR,M} - P\text{Cost}_{S}^{PR,M} \cdot \text{PROD}_{S}^{PR,M} + \lambda_{S}^{PR,M} \left( \text{PROD}_{S}^{PR,M} - \text{SALES}_{S}^{PR,M} \right)
\]

Site SUBPROBLEM for all \(S\) \((NLP)\)  
Market SUBPROBLEM for all \(M\) \((LP)\)
Temporal Decomposition

• Decompose at each time period

• Duplicate variables for Inventories for each time period

• Apply Langrangean Decomposition Algorithm
### Multisite Distribution Model - Spatial

- 3 Multi-Plant Sites, 3 Geographic Markets
- Solved with GAMS/Conopt2

<table>
<thead>
<tr>
<th># Time Periods (months)</th>
<th>Variables/Constraints</th>
<th>Optimal Solution Profit (million-$)</th>
<th>Full Space Solution Time (CPU sec)</th>
<th>Lagrangean Solution Time (CPU sec)</th>
<th>% Within Full Optimal Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3345 / 2848</td>
<td>164</td>
<td>52</td>
<td>10</td>
<td>10%</td>
</tr>
<tr>
<td>4</td>
<td>6689 / 5698</td>
<td>326</td>
<td>478</td>
<td>127</td>
<td>11%</td>
</tr>
<tr>
<td>6</td>
<td>10033 /8548</td>
<td>497</td>
<td>1605</td>
<td>279</td>
<td>9%</td>
</tr>
<tr>
<td>8</td>
<td>13377/11398</td>
<td>666</td>
<td>2350</td>
<td>550</td>
<td>9%</td>
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</table>
Multisite Distribution Model - Temporal

- 3 Multi-Plant Sites, 3 Geographic Markets
- Solved with GAMS/Conopt2

<table>
<thead>
<tr>
<th># Time Periods (months)</th>
<th>Variables/Constraints</th>
<th>Optimal Solution Profit (million-$)</th>
<th>Full Space Solution Time (CPU sec)</th>
<th>Lagrangean Solution Time (CPU sec)</th>
<th>% Within Full Optimal Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>5230 / 5005</td>
<td>116.05</td>
<td>395</td>
<td>97</td>
<td>2.2</td>
</tr>
<tr>
<td>6</td>
<td>9973 / 8551</td>
<td>236.53</td>
<td>2013</td>
<td>138</td>
<td>2.3</td>
</tr>
<tr>
<td>12</td>
<td>19945 / 17101</td>
<td>474.18</td>
<td>10254</td>
<td>278</td>
<td>2.2</td>
</tr>
</tbody>
</table>

Temporal much smaller gap!

**Reason:** material balances not violated at each time period
Stochastic Optimization for Gasfield Planning

Goel, Grossmann, El-Amr, Mulckay (2006)

Superstructure

- Certain Fields
  - Fields
  - Well Platforms (WP)
  - Connecting pipelines
  - Production Platforms (PP)

- Uncertain Fields
  - Size
  - Deliverability

Decisions

- Investment
  - Platforms: Which and When
  - Pipelines: Which and When
  - Platform Capacities

- Operational
  - Production profile: each field

Time Horizon

Discretized into time periods: 1 year each
Representation of Uncertainty

- **Discrete probability distribution**
  - Size and Initial Deliverability of each uncertain field

- **Scenario**
  - Possible combination of sizes and initial deliverabilities of different uncertain fields
  - Each scenario has a given probability

- **Objective**
  - Maximize Expected NPV
  - Expected NPV (ENPV) = \( \sum_s p_s \text{NPVs} \)

---

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Decision Dependent Scenario Trees

**Assumption:** Uncertainty in a field resolved as soon as WP installed at field

- Invest in F in year 1
- Invest in F in year 5

Scenario tree

- Not unique: Depends on timing of investment at uncertain fields
- Central to defining a Stochastic Programming Model
Novel Stochastic Programming Model (SPM)

Maximize Expected NPV

\[
\text{Max } \sum_s p^s \left[ \sum_t \left( c_{1t} q^s_t + c_{2t} d^s_t + c_{3t} y^s_t + \sum_{uf} c_{4t,uf} b_{uf,t}^s \right) \right]
\]

**Linear Investment, Operation Constraints**

- \( A^s q^s_t \leq a^s \)  \( \forall (t,s) \)
- \( g_t(q^s_1, q^s_2, \ldots, q^s_t) \leq 0 \)  \( \forall (t,s) \)
- \( h_t(d^s_1, d^s_2, \ldots, d^s_t) \leq 0 \)  \( \forall (t,s) \)
- \( \sum b_{uf,t}^s \leq 1 \)  \( \forall (uf,t) \)
- \( r^t_{uf,t}(q^s_t, d^s_t, b_{uf,1}^s, b_{uf,2}^s, \ldots, b_{uf,t}^s, y^s_1, y^s_2, \ldots, y^s_t) \leq 0 \)  \( \forall (uf,t,s) \)

**Non-Anticipativity Constraints**

- \( q^s_t \): Cont. Operation vars.
- \( d^s_t \): Cont. Investment vars.
- \( b_{uf,t}^s \): 0-1 Investment vars., \( uf \)
- \( y^s_t \): Other 0-1 investment vars.
- \( Z_t^{s,s'} \): Boolean variables

**uf**: uncertain field
**s, s'**: scenario
**t**: time period
Novel Stochastic Programming Model (SPM)

Maximize Expected NPV

\[
\text{Max } \sum_s p^s \left[ \sum_l (c_{1l} q^s_l + c_{2l} d^s_l + c_{3l} y^s_l + \sum_{uf} c_{4l,uf} b^s_{uf,l}) \right]
\]

Linear Investment, Operation Constraints

\[
\begin{align*}
A^s q^s_l &\leq a^s \\
g_t(q^s_1, q^s_2, \ldots, q^s_t) &\leq 0 \\
h_t(d^s_1, d^s_2, \ldots, d^s_t) &\leq 0 \\
z \sum_{uf} b^s_{uf,l} &\leq 1 \\
y^s_{uf,l}(q^s_1, d^s_t, b^s_{uf,1}, b^s_{uf,2}, \ldots, b^s_{uf,l}, y^s_1, y^s_2, \ldots, y^s_t) &\leq 0
\end{align*}
\]

Non-Anticipativity Constraints

uf: uncertain field
s, s': scenario
t: time period

q^s_l: Cont. Operation vars.
d^s_l: Cont. Investment vars.
b^s_{uf,l}: 0-1 Investment vars., uf
y^s_l: Other 0-1 investment vars.
Z^s,s': Boolean variables

Generalized Disjunctive Program
Branch and Bound Algorithm

**Major steps** (Maximization problem) at every node

- Upper bound: **Lagrangian dual**
- Lower bound: **Feasible solutions**
- Branching
Formulation of Lagrangean dual

**Relaxation**
- Relax disjunctions, logic constraints
- Penalty for equality constraints
  \( b_{u_f}^{s,s'}, y_{s,s'}, d_{s,s'} \)

Lagrange Multipliers

\[
\begin{align*}
\text{Max} & \quad \sum_s p^s \left[ \sum_t \left( c_{1t} q_t^s + c_{2t} d_t^s + c_{3t} y_t^s + \sum_{u_f} c_{4t,u_f} b_{u_f,t}^s \right) \\
& \quad + \sum_{(s,s')} \sum_{u_f} \lambda_{u_f}^{s,s'} (b_{u_f,1}^s - b_{u_f,1}^{s'}) + \lambda_{s,s'} (y_1^s - y_1^{s'}) + d_{s,s'} (d_1^s - d_1^{s'}) \right] \\
\sum_{\tau=1}^t \left( A_{\tau}^s q_{\tau}^s + B_{\tau}^s d_{\tau}^s + C_{\tau}^s y_{\tau}^s + \sum_{u_f} D_{u_f,\tau}^s b_{u_f,\tau}^s \right) & \leq a_t^s \quad \forall (t, s)
\end{align*}
\]
Formulation of Lagrangean dual

Relaxation $\phi(\lambda) =$
- Relax disjunctions, logic constraints
- Penalty for equality constraints $b_\lambda^{s,s'}, y_\lambda^{s,s'}, d_\lambda^{s,s'}$:

Lagrange Multipliers

$$\max \sum_s p^s \left[ \sum_t \left( c_{1t} q^s_t + c_{2t} d^s_t + c_{3t} y^s_t + \sum_{uf} c_{4t,uf} b^s_{uf,t} \right) \right]$$
$$+ \sum_s \sum_{(s,s')} \lambda^{s,s'} \left( b^{s}_{uf,1} - b^{s'}_{uf,1} \right) + y^{s,s'} \left( y^s_1 - y^{s'}_1 \right) + d^{s,s'} \left( d^s_1 - d^{s'}_1 \right)$$

$$\sum_{t=1}^t \left( A^s_{t} q^s_{t} + B^s_{t} d^s_{t} + C^s_{t} y^s_{t} + \sum_{uf} D^s_{uf,t} b^s_{uf,t} \right) \leq a^s_t \quad \forall (t, s)$$

Model decomposes: one MILP, each scenario!!!

$$\phi^s(\lambda) =$$

$$\max p^s \left[ \sum_t \left( c_{1t} q^s_t + c_{2t} d^s_t + c_{3t} y^s_t + \sum_{uf} c_{4t,uf} b^s_{uf,t} \right) \right]$$
$$+ \sum_{s'} (-1)^{k(s')} \sum_{uf} \lambda^{s,s'} b^s_{uf,1} + y^{s,s'} y^s_1 + d^{s,s'} d^s_1$$

$$\sum_{t=1}^t \left( A^s_{t} q^s_{t} + B^s_{t} d^s_{t} + C^s_{t} y^s_{t} + \sum_{uf} D^s_{uf,t} b^s_{uf,t} \right) \leq a^s_t \quad \forall t$$

Lagrangean Relaxation: Upper bound for any $\lambda$

$$\sum_s \phi^s(\lambda) = \phi(\lambda) \geq \phi_{optimal} \quad \forall \lambda$$

Lagrangean Dual: Tightest upper bound

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$$\phi_{LD} = \min_{\lambda} \phi(\lambda) \geq \phi_{optimal}$$
Example

15 time periods

Uncertainty sizes of fields B and C: 9 scenarios

Size MILP Model:
16,281 0-1, 125,956 cont var, 386,597 const

Stochastic programming solution

<table>
<thead>
<tr>
<th>Scenario</th>
<th>1</th>
<th>4</th>
<th>7</th>
<th>2</th>
<th>5</th>
<th>8</th>
<th>3</th>
<th>6</th>
<th>9</th>
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</thead>
<tbody>
<tr>
<td>$t = 1$</td>
<td>A(66.68), B(83.32), PP(150.00)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t = 6$</td>
<td>C(73.86)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t = 7$</td>
<td>C(73.33)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t = 8$</td>
<td>C(73.33)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t = 10$</td>
<td>D(66.26)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t = 12$</td>
<td>D(69.75)</td>
<td>D(70.54)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t = 13$</td>
<td>D(49.01)</td>
<td>D(55.18)</td>
<td>D(60.99)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ENPV</td>
<td>$65.770$ Million</td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

$65.770$ Million Stochastic vs Deterministic solution: $61.215$ Million
Branch and Bound Tree

9 nodes, 1683 CPU sec (1% optimality gap)
Concluding Remarks

1. Lagrangean relaxation/decomposition is well established method for solving large-scale MILPs, MINLPs, NLPs

2. Non-obvious part is how to apply it to effectively decompose a problem

Some guidelines:
- Use Lagrangean decomposition, not Lagrangean relaxation
- Avoid if possible relaxing “critical” constraints (eg mass balances)
- Try to decompose so that effect of relaxed equations is not too large