GLOBAL OPTIMIZATION WITH BRANCH-AND-REDUCE

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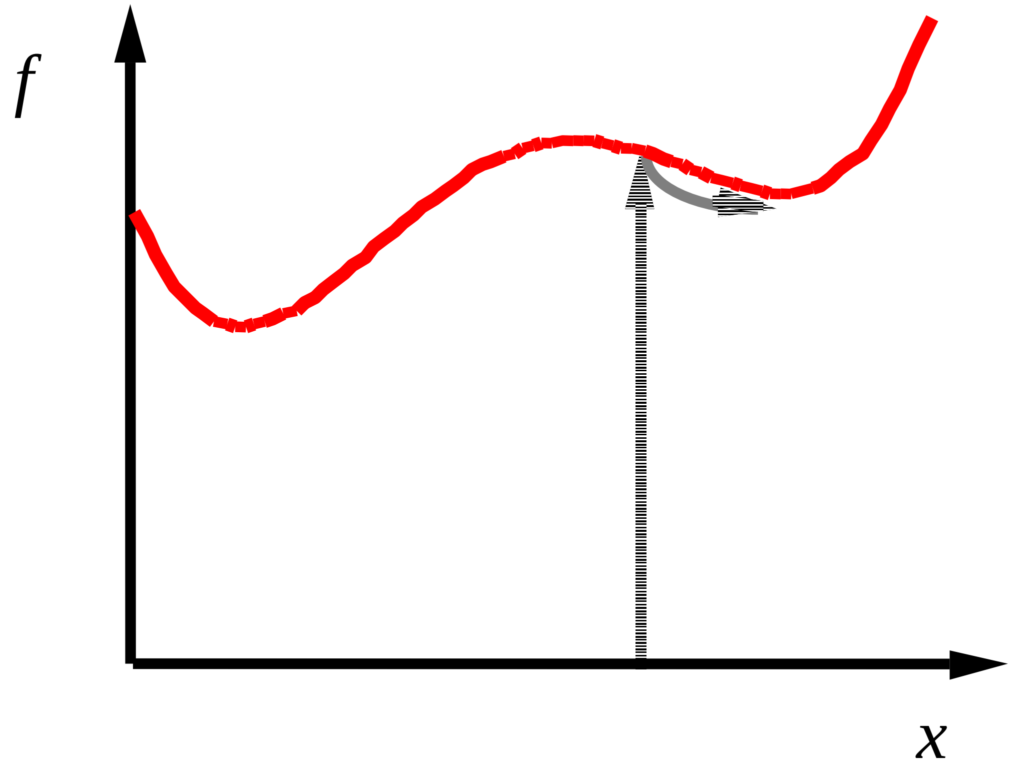
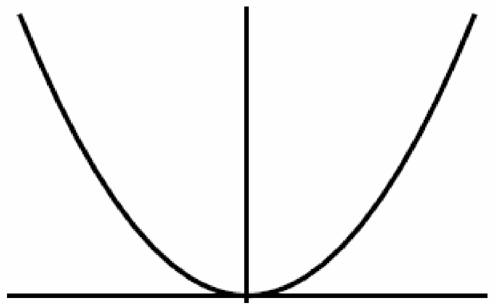
THE MULTIPLE-MINIMA DIFFICULTY IN OPTIMIZATION

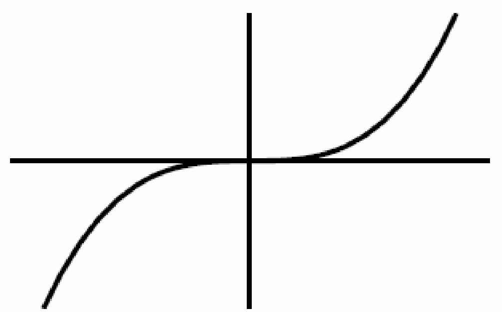


- **Classical optimality conditions are necessary but not sufficient**
- **Classical optimization provides the local minimum "closest" to the starting point used**

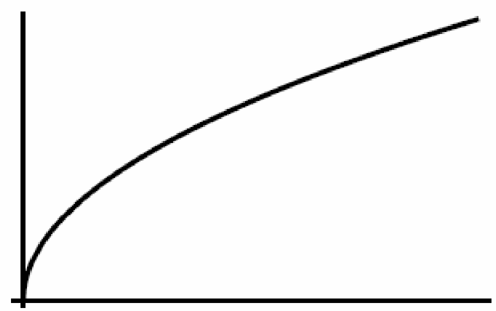
COMMON FUNCTIONS IN MODELING



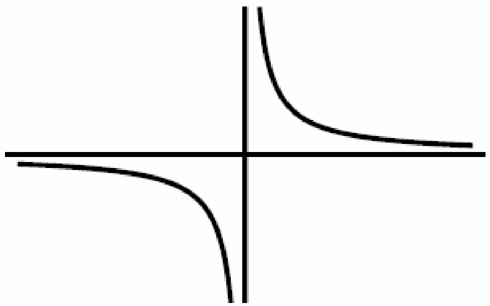
x^2



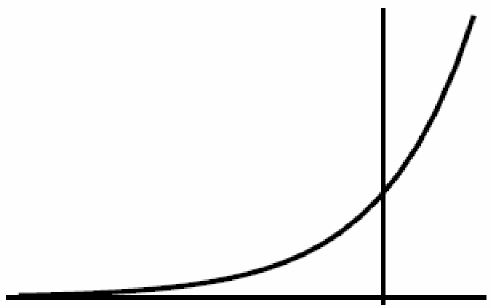
x^3



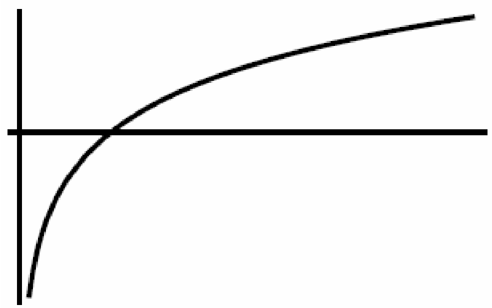
$\sqrt{x}$



$\frac{1}{x}$

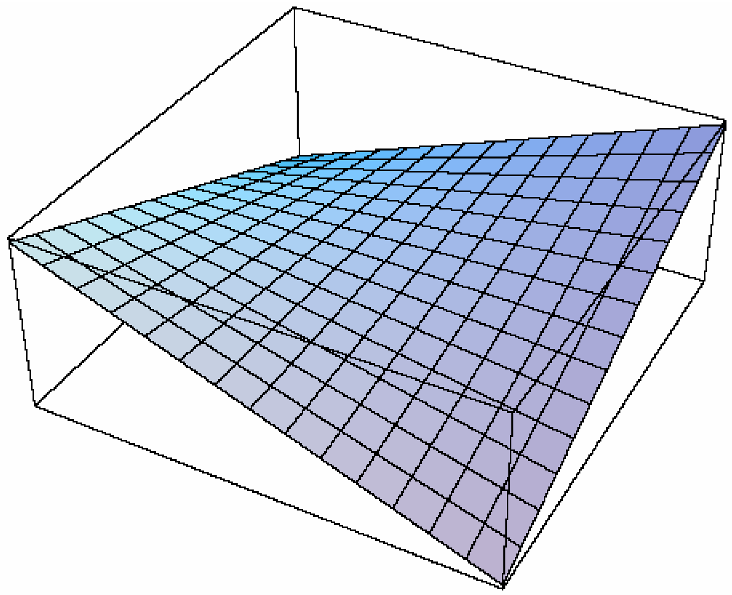


$\exp(x)$

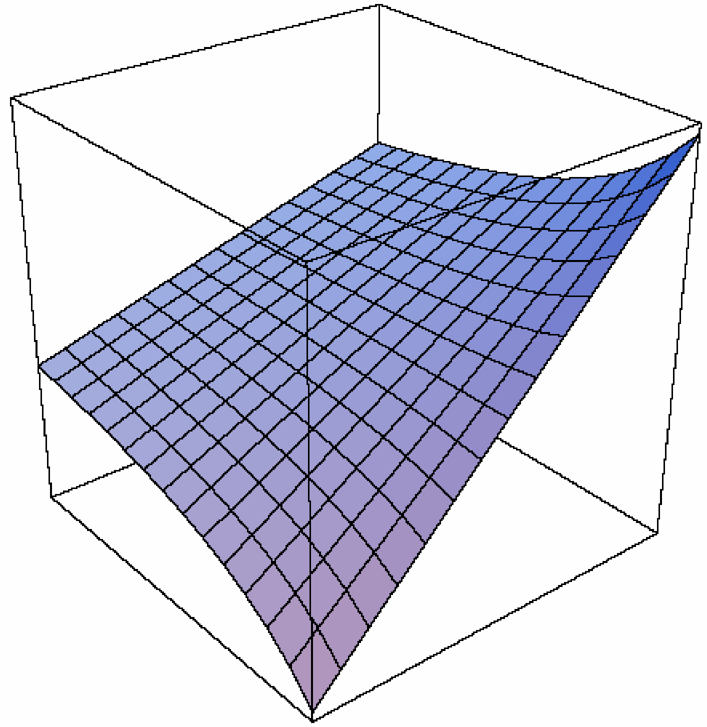


$\log(x)$

COMMON FUNCTIONS IN MODELING

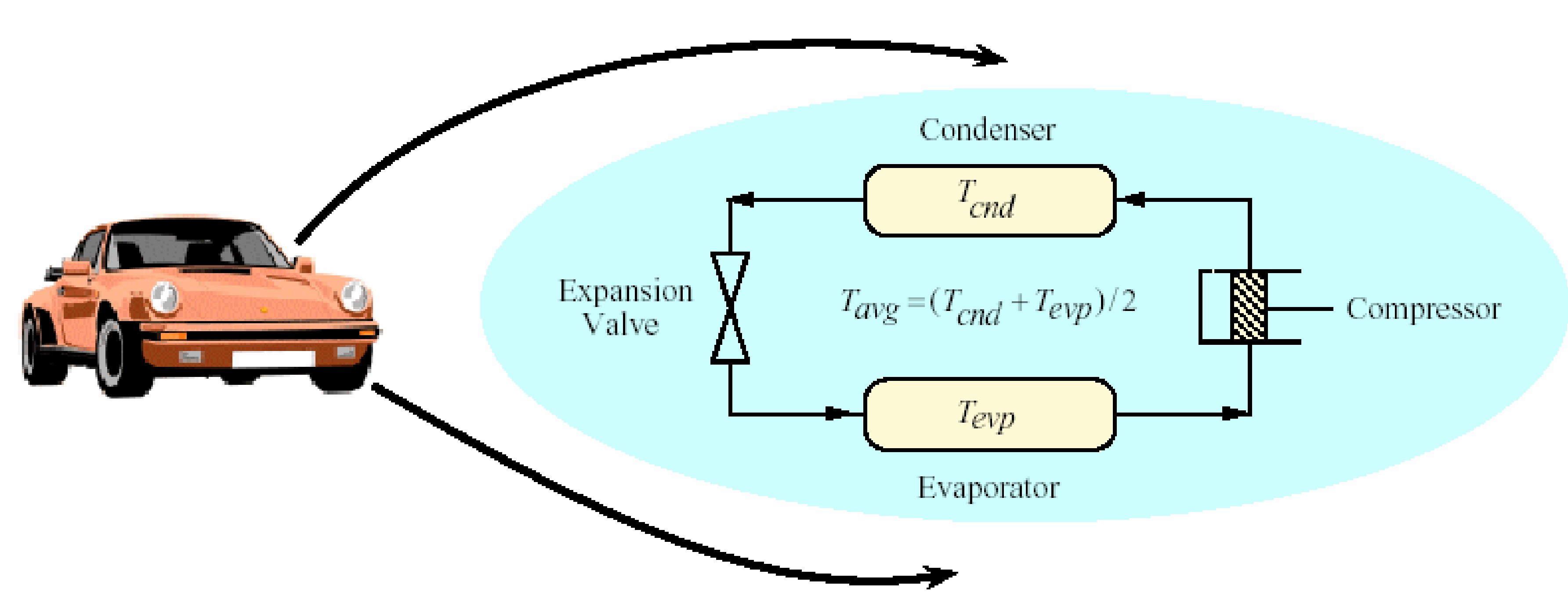


$x1 * x2$



$\frac{x1}{x2}$

AUTOMOTIVE REFRIGERANT DESIGN
(Joback and Stephanopoulos, 1990)



- **Higher enthalpy of vaporization (ΔH_{ve}) reduces the amount of refrigerant**
- **Lower liquid heat capacity (C_{pla}) reduces amount of vapor generated in expansion valve**
- **Maximize $\Delta H_{ve}/ C_{pla}$, subject to: $\Delta H_{ve} \geq 18.4$, $C_{pla} \leq 32.2$**

FUNCTIONAL GROUPS CONSIDERED

Acyclic Groups	Cyclic Groups	Halogen Groups	Oxygen Groups	Nitrogen Groups	Sulfur Groups
$-CH_3$	$^{r} - CH_2 - ^{r}$	$-F$	$-OH$	$-NH_2$	$-SH$
$-CH_2-$	$^{r}_{r} > CH - ^{r}$	$-Cl$	$-O-$	$> NH$	$-S-$
$> CH-$	$_{r} > CH - ^{r}$	$-Br$	$^{r} - O - ^{r}$	$^{r}_{r} > NH$	$^{r} - S - ^{r}$
$> C <$	$^{r}_{r} > C < ^{r}_{r}$	$-I$	$> CO$	$> N-$	
$= CH_2$	$_{r} > C < ^{r}_{r}$		$^{r}_{r} > CO$	$= N-$	
$= CH-$	$> C < ^{r}_{r}$		$-CHO$	$^{r} = N - ^{r}$	
$= C <$	$^{r} = CH - ^{r}$		$-COOH$	$-CN$	
$= C =$	$^{r} = C < ^{r}_{r}$		$-COO-$	$-NO_2$	
$\equiv CH$	$^{r} = C < _{r}$		$= O$		
$\equiv C-$	$= C < ^{r}_{r}$				

Number of Groups = 44

Maximum Selection Size = 15

Candidates = 39,895,566,894,524

PROPERTY PREDICTION

$$T_b = 198.2 + \sum_{i=1}^{N} n_i T_{bi}$$

$$T_c = \frac{T_b}{0.584 + 0.965 \sum_{i=1}^{N} n_i T_{ci} - (\sum_{i=1}^{N} n_i T_{ci})^2}$$

$$P_c = \frac{1}{(0.113 + 0.0032 \sum_{i=1}^{N} n_i a_i - \sum_{i=1}^{N} n_i P_{ci})^2}$$

$$C_{p0a} = \sum_{i=1}^{N} n_i C_{p0ai} - 37.93 + \left(\sum_{i=1}^{N} n_i C_{\text{p0bi}} + 0.21\right) T_{\text{avg}} + \left(\sum_{i=1}^{N} n_i C_{p0ci} - 3.91 \times 10^{-4}\right) T_{\text{avg}}^2 + \left(\sum_{i=1}^{N} n_i C_{\text{p0di}} + 2.06 \times 10^{-7}\right) T_{\text{avg}}^3$$

$$T_{\text{br}} = \frac{T_{\text{b}}}{T_c}$$

$$T_{\text{avgr}} = \frac{T_{\text{avg}}}{T_c}$$

$$T_{\text{cndr}} = \frac{T_{\text{cnd}}}{T_c}$$

$$T_{\text{evpr}} = \frac{T_{\text{evp}}}{T_c}$$

$$\alpha = -5.97214 - \ln\left(\frac{P_c}{1.013}\right) + \frac{6.09648}{T_{\text{br}}} + 1.28862 \ln(T_{\text{br}}) - 0.169347 T_{\text{br}}^6$$

$$\beta = 15.2518 - \frac{15.6875}{T_{\text{br}}} - 13.4721 \ln(T_{\text{br}}) + 0.43577 T_{\text{br}}^6$$

$$\omega = \frac{\alpha}{\beta}$$

$$C_{\text{pla}} = \frac{1}{4.1868} \left\{ C_{\text{p0a}} + 8.314 \left[1.45 + \frac{0.45}{1 - T_{\text{avgr}}} + 0.25\omega \left(17.11 + 25.2 \frac{(1 - T_{\text{avgr}})^{1/3}}{T_{\text{avgr}}} + \frac{1.742}{1 - T_{avgr}} \right) \right] \right\}$$

$$\Delta H_{\text{vb}} = 15.3 + \sum_{i=1}^{N} n_i \Delta H_{\text{vbi}}$$

$$\Delta H_{\text{ve}} = \Delta H_{\text{vb}} \left(\frac{1 - T_{\text{evp}}/T_c}{1 - T_b/T_c} \right)^{0.38}$$

$$h = \frac{T_{br} \ln(P_c/1.013)}{1 - T_{br}}$$

$$G = 0.4835 + 0.4605 h$$

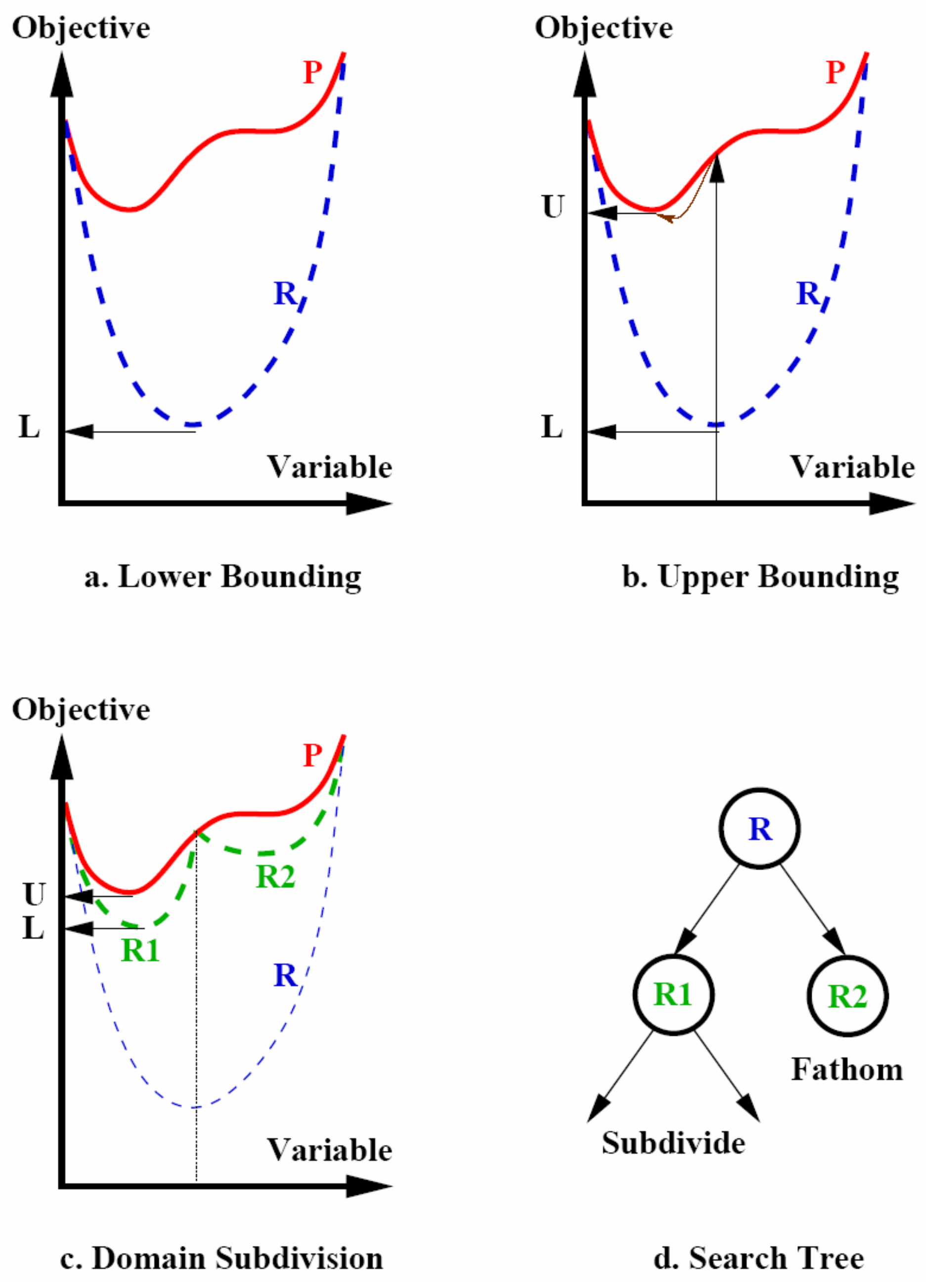
$$k = \frac{h/G - (1 + T_{br})}{(3 + T_{br})(1 - T_{br})^2}$$

$$\ln P_{\text{vpcr}} = \frac{-G}{T_{\text{cndr}}} \left[1 - T_{\text{cndr}}^2 + k(3 + T_{\text{cndr}})(1 - T_{\text{cndr}})^3 \right]$$

$$\ln P_{\text{vper}} = \frac{-G}{T_{\text{evpr}}} \left[1 - T_{\text{evpr}}^2 + k(3 + T_{\text{evpr}})(1 - T_{\text{evpr}})^3 \right]$$

n_i integer

BRANCH-AND-BOUND



MOLECULAR DESIGN AFTER 150 CPU HOURS IN 1995

- One feasible solution identified
- Optimality not proved

- First attempt:
  - IBM RS/6000 43P with 128 MB RAM
- Second attempt:
  - IBM SP/2 Single Processor with 2 GB RAM

MOLECULAR DESIGN IN 2000

	Molecular Structure	$\frac{\Delta H_{ve}}{C_{pla}}$
FNO	$F-N=O$	1.2880
FSH	$F-SH$	1.1697
CH_3Cl	CH_3-Cl	1.1219
ClFO	$(Cl-)(-O-)(-F)$	0.9822
C_2HClO_2	$O=C<(-CH=O)(-Cl)$	1.1207
C_3H_40	$CH_3-CH=C=O$	0.9619
C_3H_4	$CH_3-C\equiv CH$	0.9278
C_2F_2	$F-C\equiv C-F$	0.9229
CH_2ClF	$F-CH2-Cl$	0.9202
C_2HO_2F	$F-O-CH=C=O$	0.8705
C_3H_4	$CH_2=C=CH_2$	0.8656
C_2H_6	CH_3-CH_3	0.8632
C_3H_3FO	$(F-)(CH3-)>C=C=O$	0.8531
NHF_2	$F-NH-F$	0.8468
C_2HOF	$CH\equiv C-O-F$	0.8263

	Molecular Structure	$\frac{\Delta H_{ve}}{C_{pla}}$
C_3H_3F	$CH\equiv C-CH_2-F$	0.7802
CHF_2Cl	$(F-)(F-)>CH-Cl$	0.7770
C_2H_3OF	$CH_2=CH-O-F$	0.7685
NF_2Cl	$(F-)(F-)>N-Cl$	0.7658
C_2H_6NF	$(CH3-)(CH3-)>N-F$	0.6817
N_2HF_3	$(F-)(F-)>N-NH-F$	0.6711
$C_2H_2OF_2$	$CH2=C<(-O-F)(-F)$	0.6705
$C_3H_2F_2$	$(F-)(F-)>CH-C\equiv CH$	0.6686
C_2HNF_2	$CH\equiv C-N<(-F)(-F)$	0.6587
$C_3H_4F_2$	$(F-)(F-CH_2-)>C=CH_2$	0.6377
$C_3H_4F_2$	$(F-)(F-)>CH-CH=CH_2$	0.6263
$C_2H_3NF_2$	$CH2=CH-N<(-F)(-F)$	0.6176
CH_3NOF_2	$(F-)(CH_3-)>N-O-F$	0.6139
$C_3H_3F_3$	$(_r> CH-\ ')_3(-F)_3$	0.5977

For CCl_2F_2, $\Delta H_{ve}/C_{pla} \approx 0.57$

In 30 CPU minutes

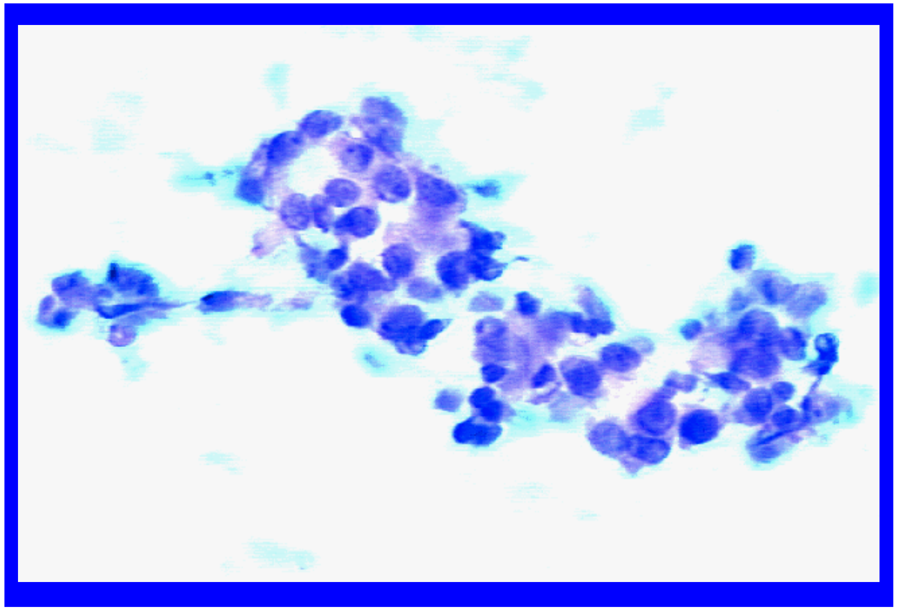
BREAST CANCER DIAGNOSIS

- **200,000 cases diagnosed in the U.S. a year**
- **40,000 deaths a year**

- **Most breast cancers are first diagnosed by the patient as a lump in the breast**
- **Majority of breast lumps are benign**

- **Available diagnosis methods:**
  - **Mammography (68% to 79% correct)**
  - **Surgical biopsy (100% correct but invasive and costly)**
  - **Fine needle aspirate (FNA)**
    - » **With visual inspection: 65% to 98% correct**
    - » **Automated diagnosis: 95% correct**
      - **Linear programming techniques**
      - **Mangasarian and Wolberg in 1990s**

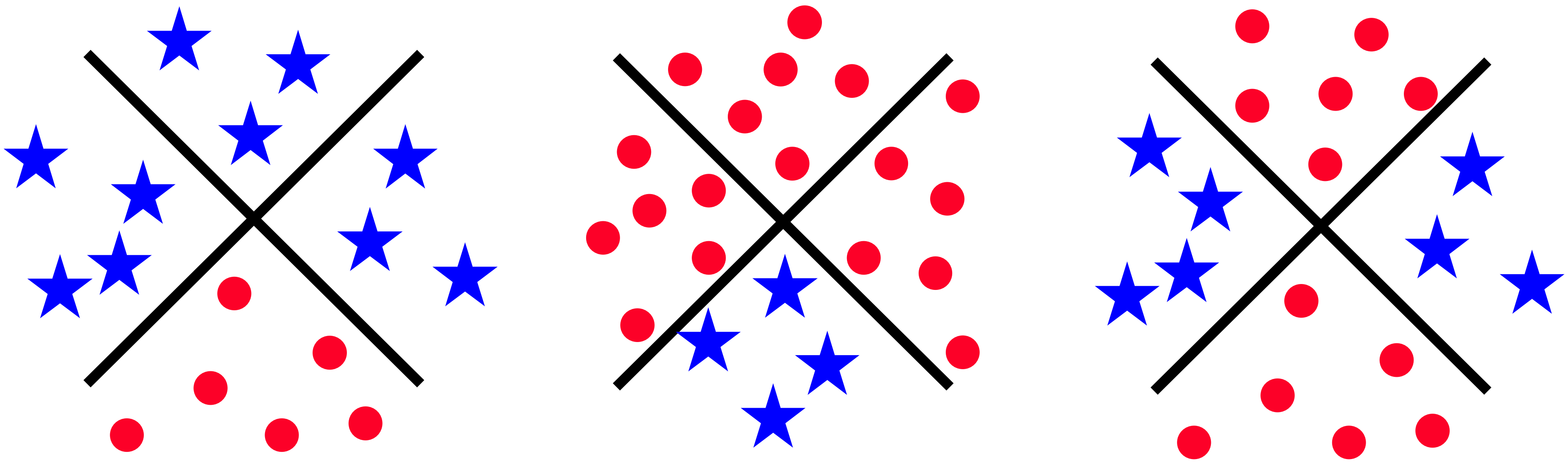
WISCONSIN DIAGNOSTIC BREAST CANCER (WDBC) DATABASE



From Wolberg, Street, & Mangasarian, 1993

- **653 patients**
- **9 cytological characteristics:**
  - Clump thickness
  - Uniformity of cell size
  - Uniformity of cell shape
  - Marginal adhesion
  - Single epithelial cell size
  - Bare nuclei
  - Bland chromatin
  - Normal nucleoli
  - Mitoses
- **Biopsy classified these 653 patients in two classes:**
  - Benign
  - Malignant

BILINEAR (IN-)SEPARABILITY OF TWO SETS IN R^n

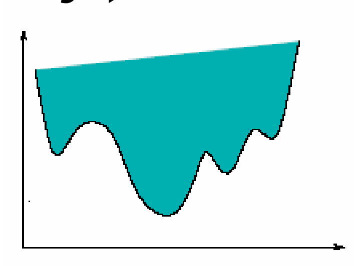


Requires the solution of three nonconvex bilinear programs

CHALLENGES IN GLOBAL OPTIMIZATION

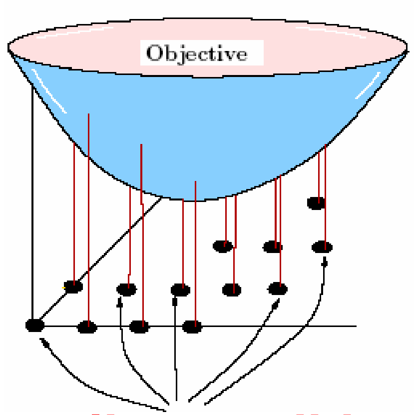
$$\begin{aligned} \min \quad & f(x,y) \\ \text{s.t.} \quad & g(x,y) \le 0 \\ & x \in R^n, \;\; y \in Z^p \end{aligned}$$

$f(x,y)$



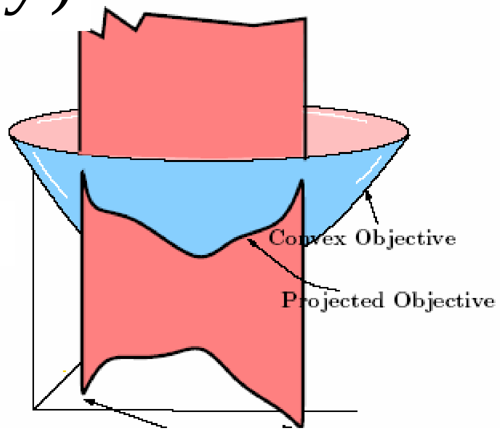
Multimodal objective

$f(x,y)$



Integrality conditions

$f(x,y)$



Nonconvex constraints

NP-HARD PROBLEM

GLOBAL OPTIMIZATION ALGORITHMS

- **Stochastic and deterministic algorithms**
- **Branch-and-Bound**
  - **Bound problem over successively refined partitions**
    - » **Falk and Soland, 1969**
    - » **McCormick, 1976**
- **Convexification**
  - **Outer-approximate with increasingly tighter convex programs**
  - **Tuy, 1964**
  - **Sherali and Adams, 1994**
- **Horst and Tuy, *Global Optimization: Deterministic Approaches*, 1996**
  - **Over 800 citations**

- **Our approach**
  - **Branch-and-Reduce**
    - » **Ryoo and Sahinidis, 1995, 1996**
    - » **Shectman and Sahinidis, 1998**
  - **Constraint Propagation & Duality-Based Reduction**
    - » **Ryoo and Sahinidis, 1995, 1996**
    - » **Tawarmalani and Sahinidis, 2002**
  - **Convexification**
    - » **Tawarmalani and Sahinidis, 2001, 2002, 2004, 2005**
- **Tawarmalani and Sahinidis, *Convexification and Global Optimization in Continuous and Mixed-Integer Nonlinear Programming*, 2002**

BOUNDING SEPARABLE PROGRAMS

$$\min \quad f = -x_1 - x_2$$

$$\text{s.t.} \quad x_3^2 - x_1^2 - x_2^2 \leq 8$$

$$x_3 = x_1 + x_2$$

$$0 \leq x_1 \leq 6$$

$$0 \leq x_2 \leq 4$$

$$0 \leq x_3 \leq 10$$

BOUNDING SEPARABLE PROGRAMS

$$\min \quad f = -x_1 - x_2$$

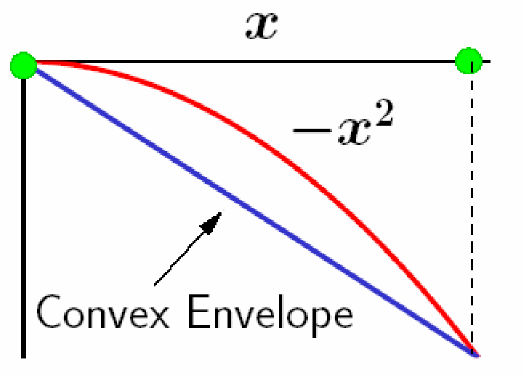
$$\text{s.t.} \quad x_3^2 - x_1^2 - x_2^2 \leq 8$$

$$x_3 = x_1 + x_2$$

$$0 \leq x_1 \leq 6$$

$$0 \leq x_2 \leq 4$$

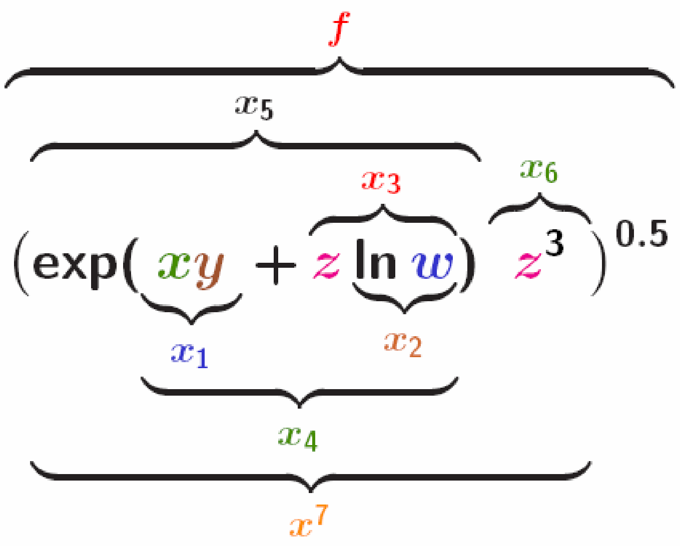
$$0 \leq x_3 \leq 10$$



BOUNDING FACTORABLE PROGRAMS

Introduce variables for intermediate quantities whose envelopes are not known

$$f(x, y, z, w) = \sqrt{\exp(xy + z \ln w) z^3}$$



$$x_1 = xy$$

$$x_2 = \ln(w)$$

$$x_3 = z x_2$$

$$x_4 = x_1 + x_3$$

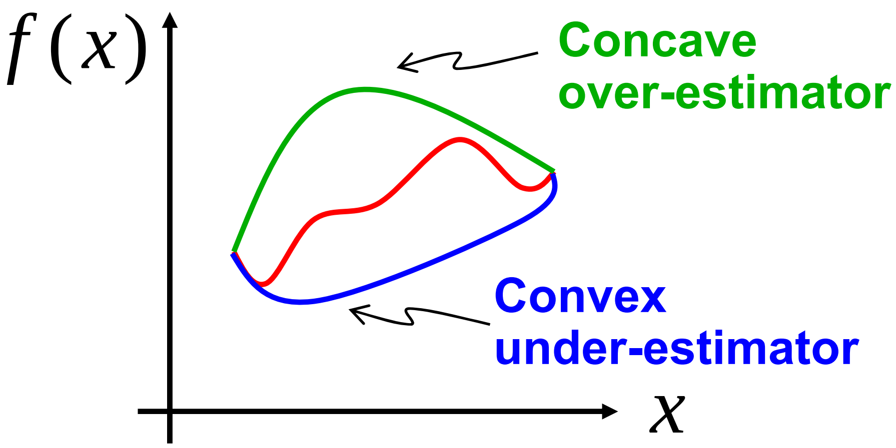
$$x_5 = \exp(x_4)$$

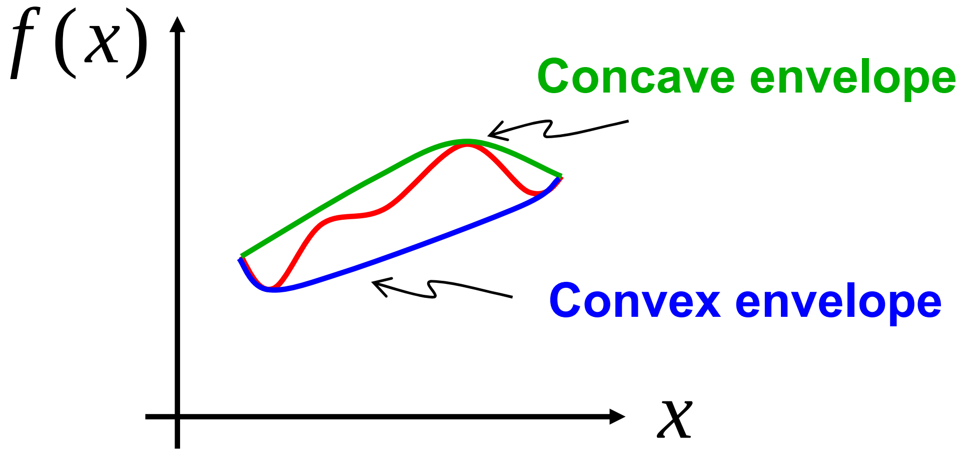
$$x_6 = z^3$$

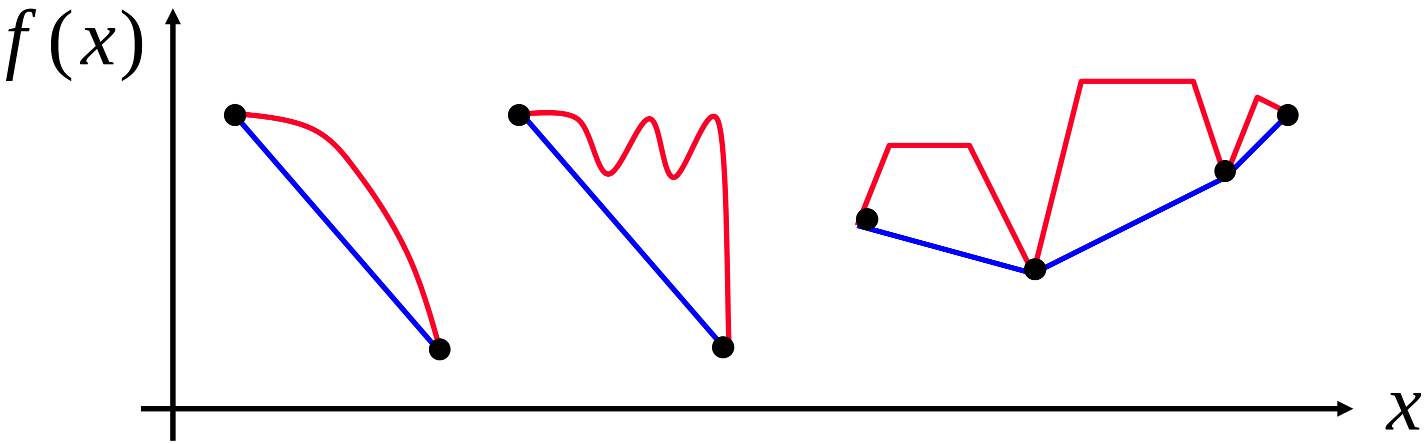
$$x_7 = x_5 x_6$$

$$f = \sqrt{x_7}$$

TIGHT RELAXATIONS

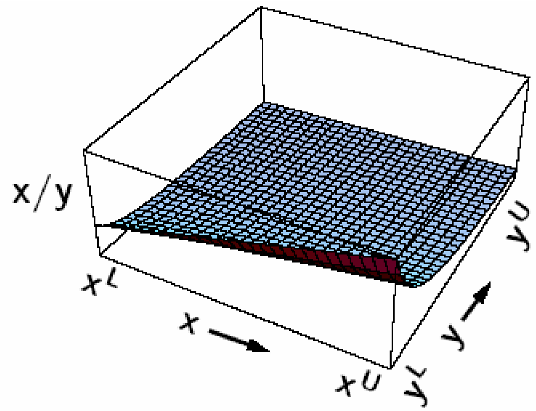






Convex/concave envelopes often finitely generated

RATIO: TRADITIONAL RELAXATION



$$z \geq x/y$$
$$y^L \leq y \leq y^U$$
$$x^L \leq x \leq x^U$$

$$z \geq x/y$$
$$x^L \leq x \leq x^U$$
$$y^L \leq y \leq y^U$$

cross-multiplying →

$$zy \geq x$$
$$y^L \leq y \leq y^U$$
$$x^L/y^U \leq z \leq x^U/y^L$$
$$x^L \leq x \leq x^U$$

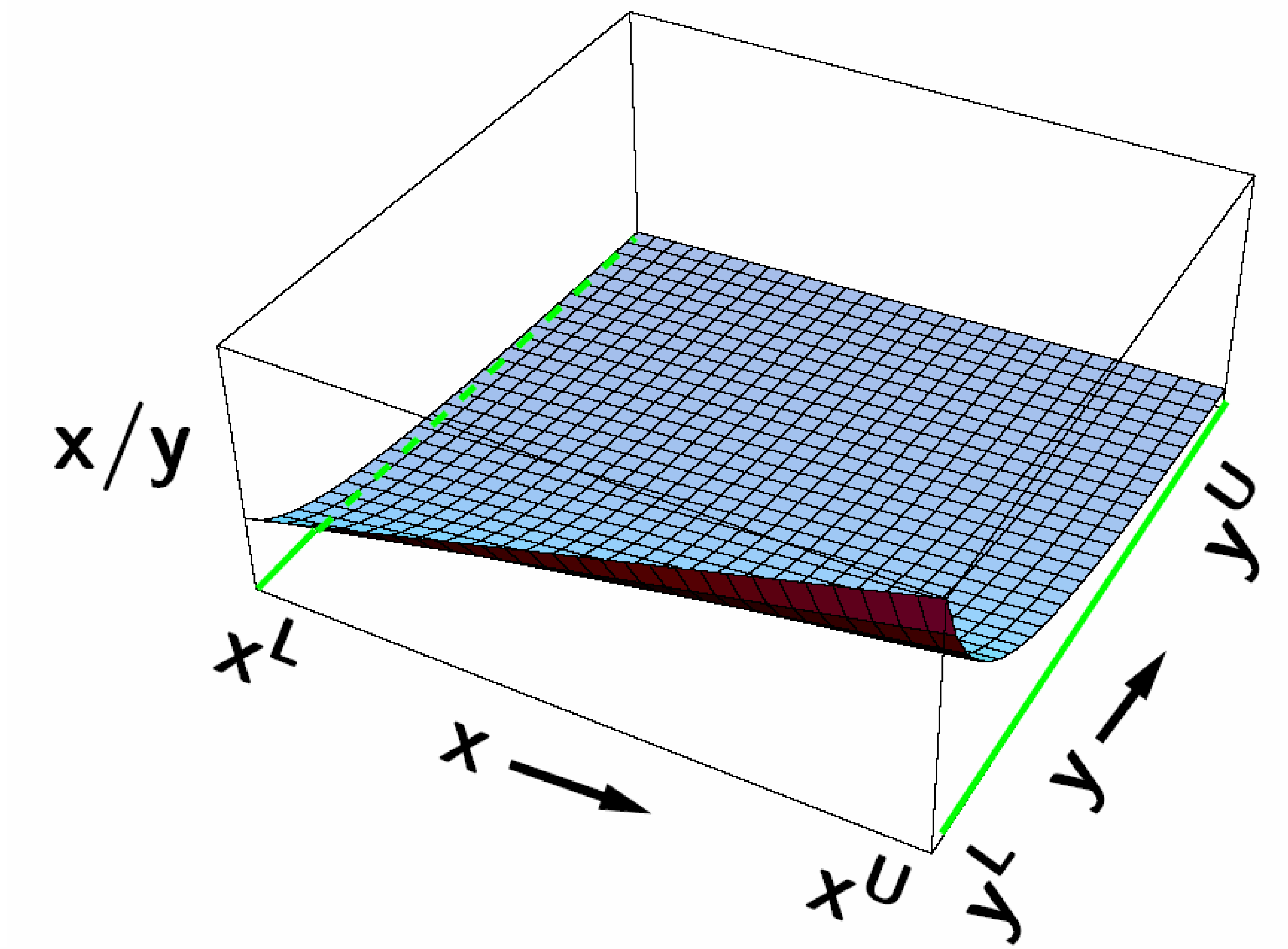
Relaxing ↓

$$zy - (z - x^L/y^U)(y - y^U) \geq x$$
$$zy - (z - x^U/y^L)(y - y^L) \geq x$$
$$y^L \leq y \leq y^U$$
$$x^L \leq x \leq x^U$$

← Simplifying

$$z \geq (xy^U - yx^L + x^L y^U)/{y^U}^2$$
$$z \geq (xy^L - yx^U + x^U y^L)/{y^L}^2$$
$$y^L \leq y \leq y^U$$
$$x^L \leq x \leq x^U$$

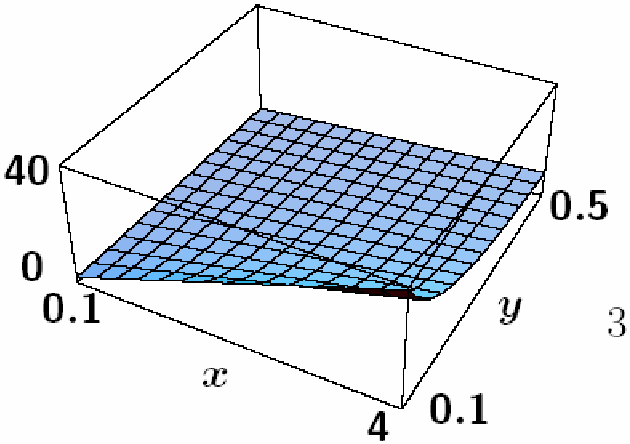
RATIO: THE GENERATING SET



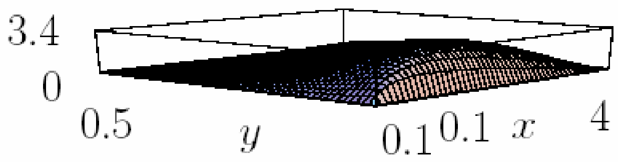
$$G^{\mathsf{epi}}(x/y) = \big\{(x,y) \;\big|\; x \in \{x^L, x^U\}\big\}$$

DIFFERENCE BETWEEN ENVELOPE AND TRADITIONAL RELAXATION

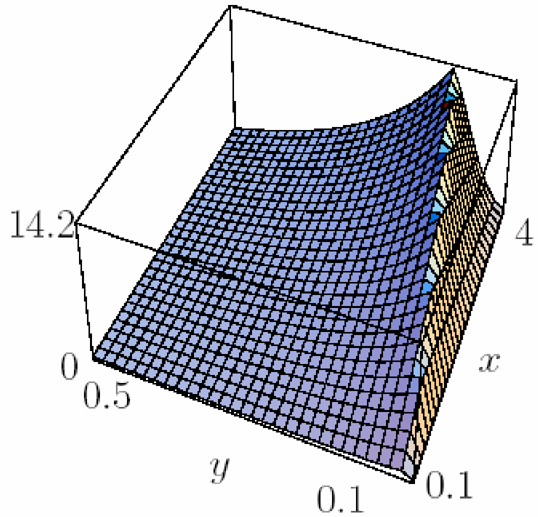
Comparison of Tightness:



Ratio: x/y



x/y − Envelope



x/y − **Traditional**

ENVELOPES OF MULTILINEAR FUNCTIONS

- **Multilinear function over a box**

$$M(x_1,\ldots,x_n) = \sum_t a_t \prod_{i=1}^{p_t} x_i, \; -\infty < L_i \le x_i \le U_i < +\infty, \; i = 1,\ldots,n$$

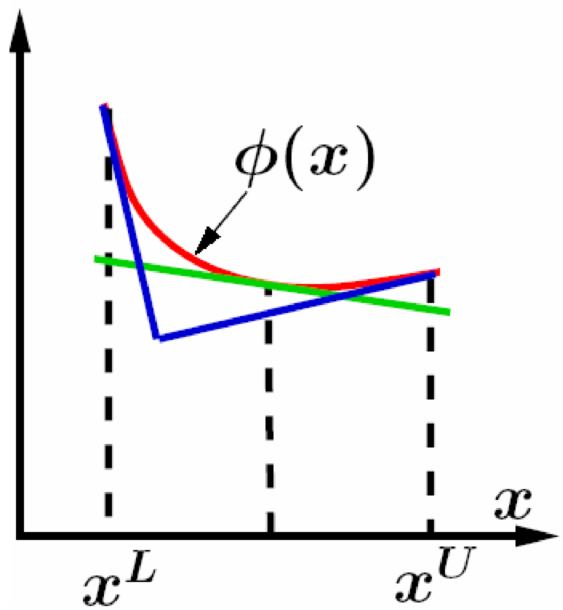
- **Generating set**

$$\text{vert}\left(\prod_{i=1}^{n} [L_i, U_i]\right)$$

- **Polyhedral convex encloser follows trivially from polyhedral representation theorems**

POLYHEDRAL OUTER-APPROXIMATION

- **Local NLP solvers essential for local search**
- **Linear programs can be solved very efficiently**
- **Outer-approximate convex relaxation by polyhedron**



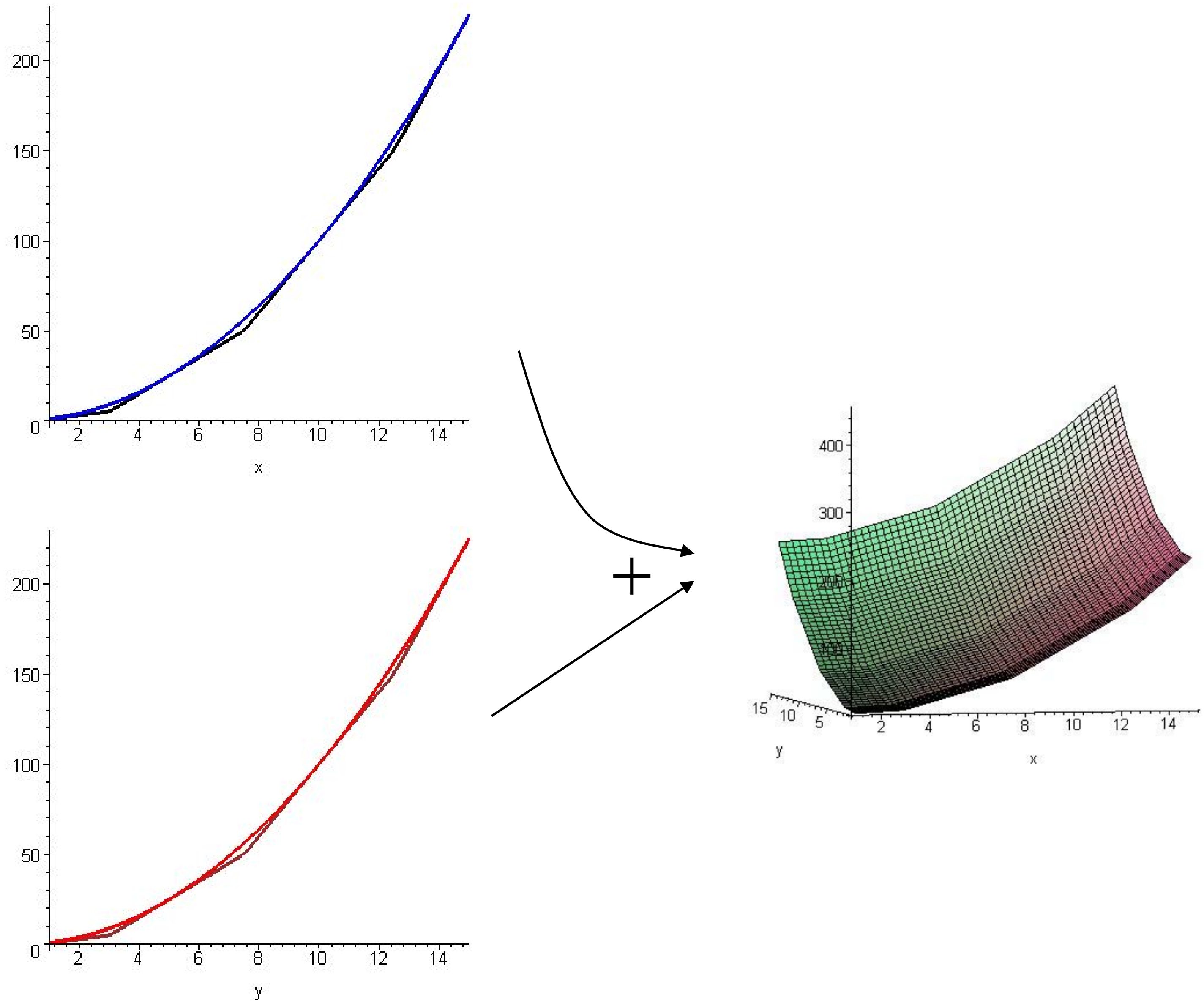
Tawarmalani and Sahinidis (*Math. Progr.*, 2004, 2005)

- **Quadratically convergent sandwich algorithm**
- **Cutting planes for functional compositions**

RECURSIVE FUNCTIONAL COMPOSITIONS

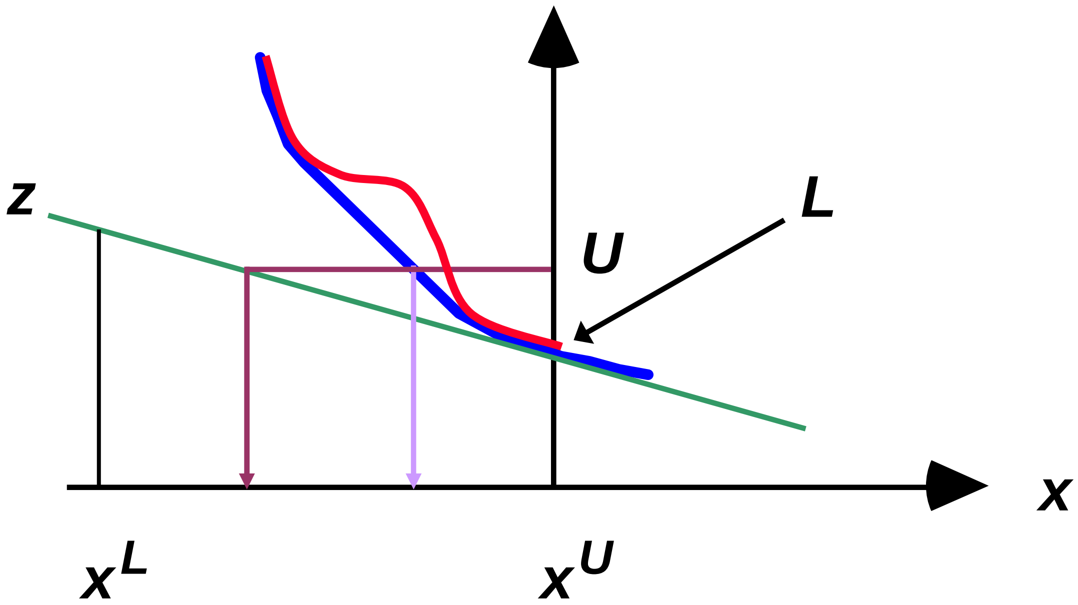
- **Consider *h*=*g*(*f*), where**
  - ***g*** **and *f* are multivariate convex functions**
  - ***g*** **is non-decreasing in the range of each nonlinear component of *f***
- ***h*** **is convex**
- **Two outer approximations of the composite function *h*:**
  - **S1: a single-step procedure that constructs supporting hyperplanes of *h* at a predetermined number of points**
  - **S2: a two-step procedure that constructs supporting hyperplanes for *g* and *f* at corresponding points**
- **Two-step is sharper than one-step**
  - **If *f* is affine, S2=S1**
  - **In general, the inclusion is strict**

OUTER APPROXIMATION OF x^2+y^2



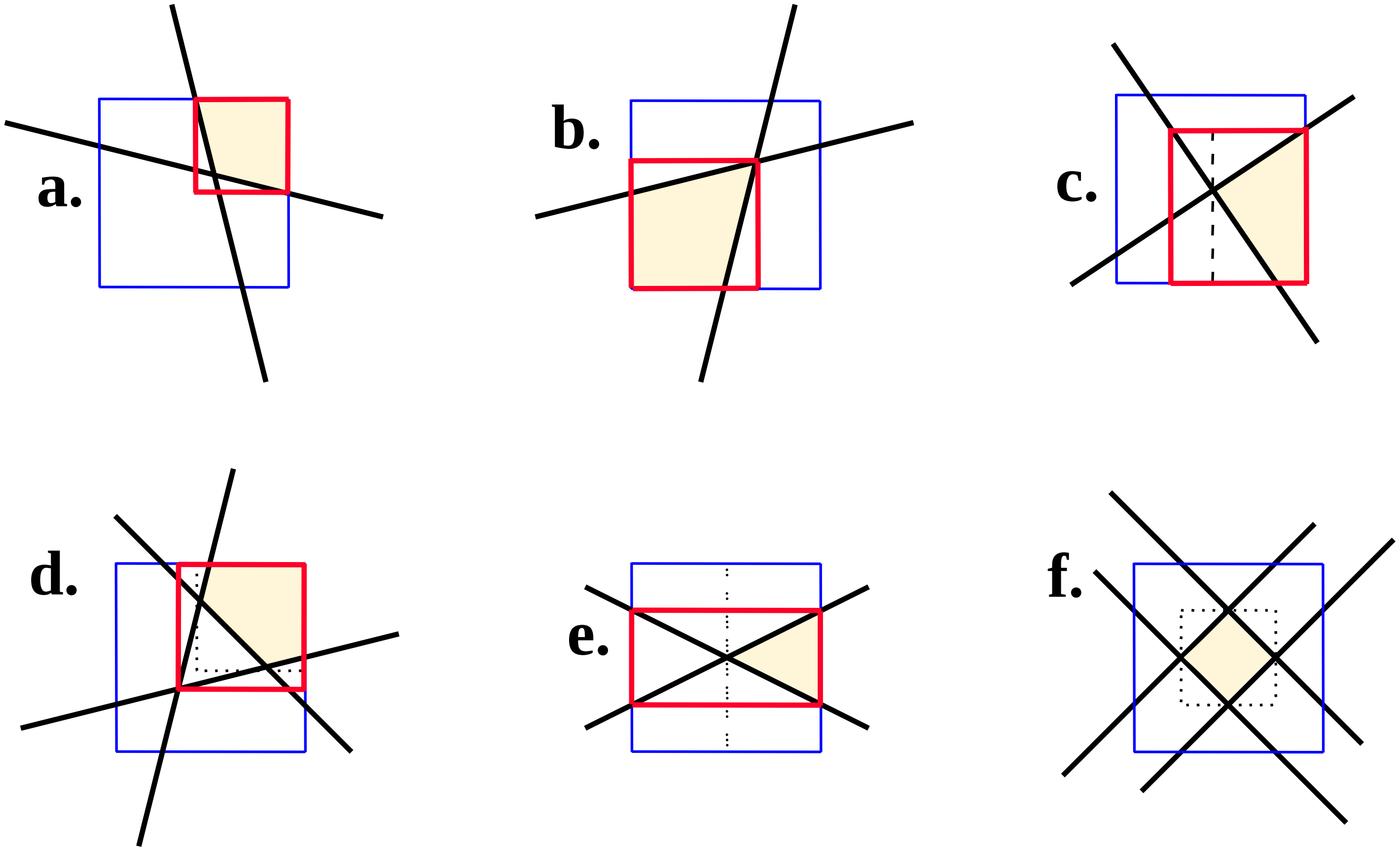
MARGINALS-BASED RANGE REDUCTION

Relaxed Value Function

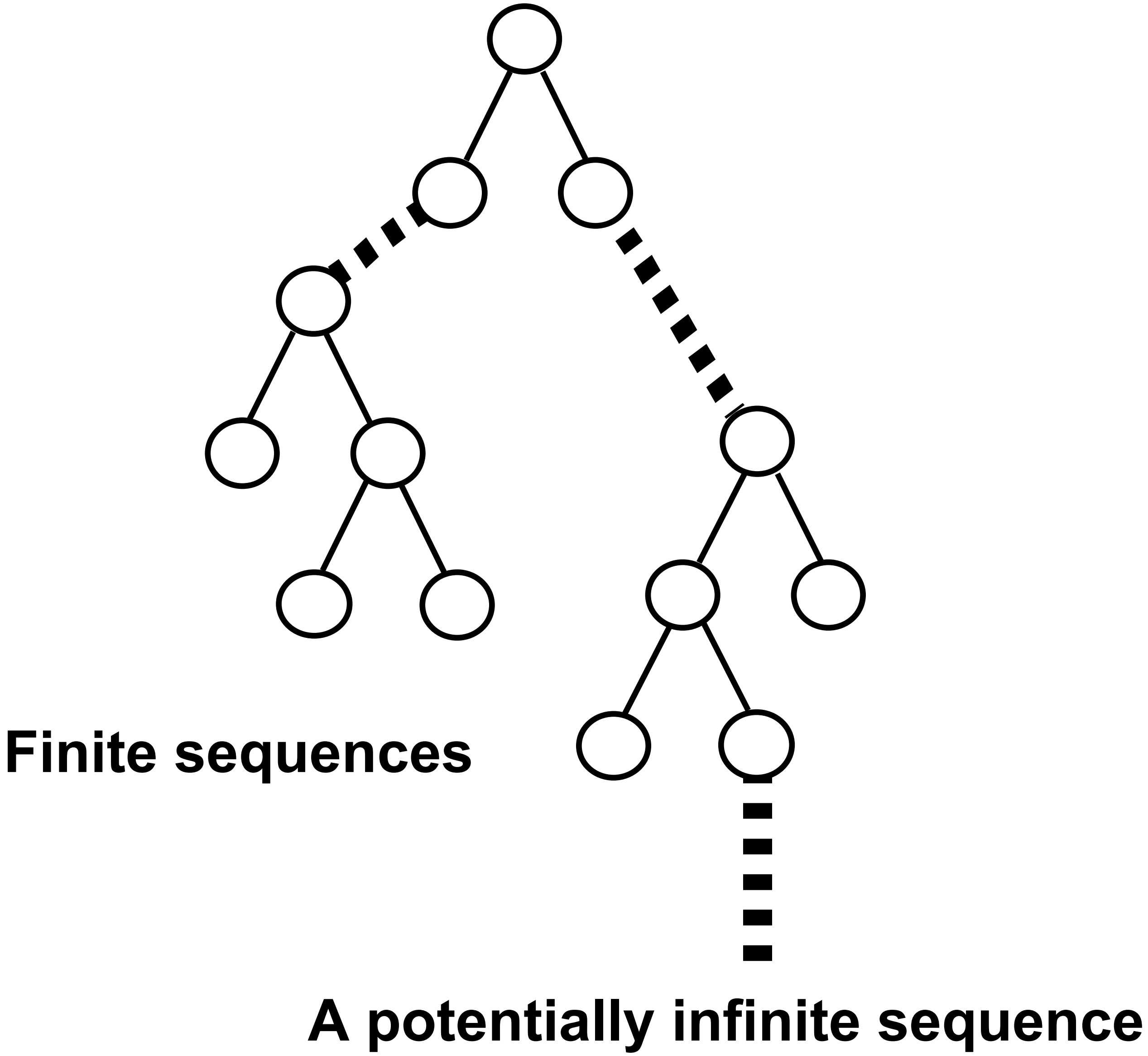


If a variable goes to its upper bound at the relaxed problem solution, this variable's lower bound can be improved

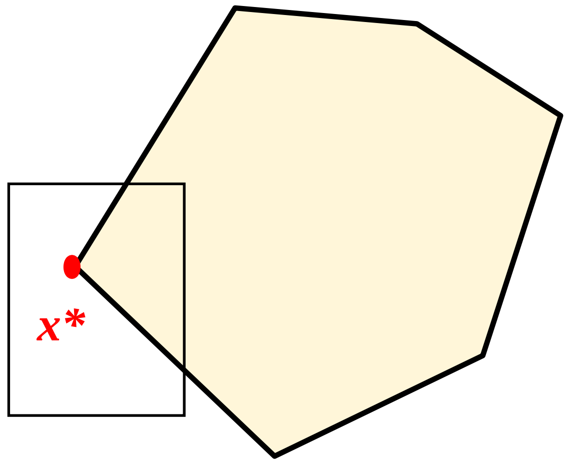
REDUCTION VIA CONSTRAINT PROPAGATION

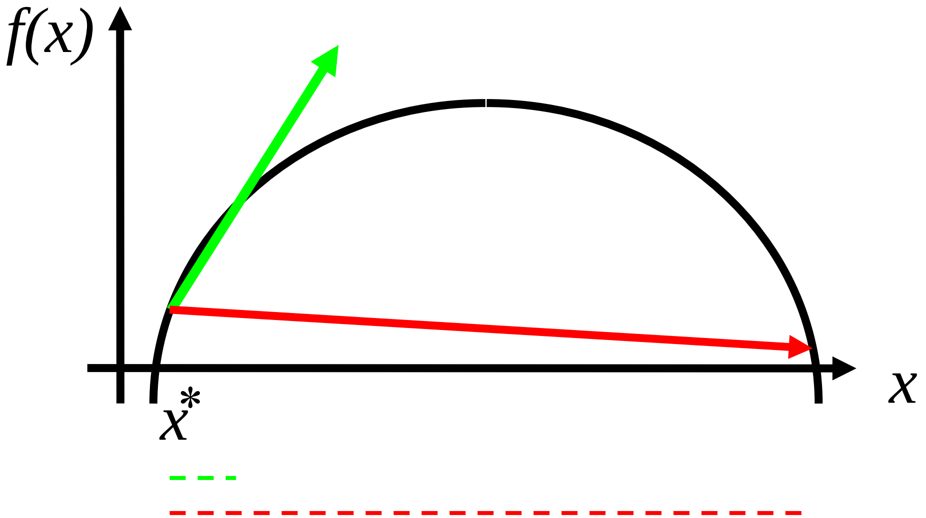


FINITE VERSUS CONVERGENT BRANCH-AND-BOUND ALGORITHMS



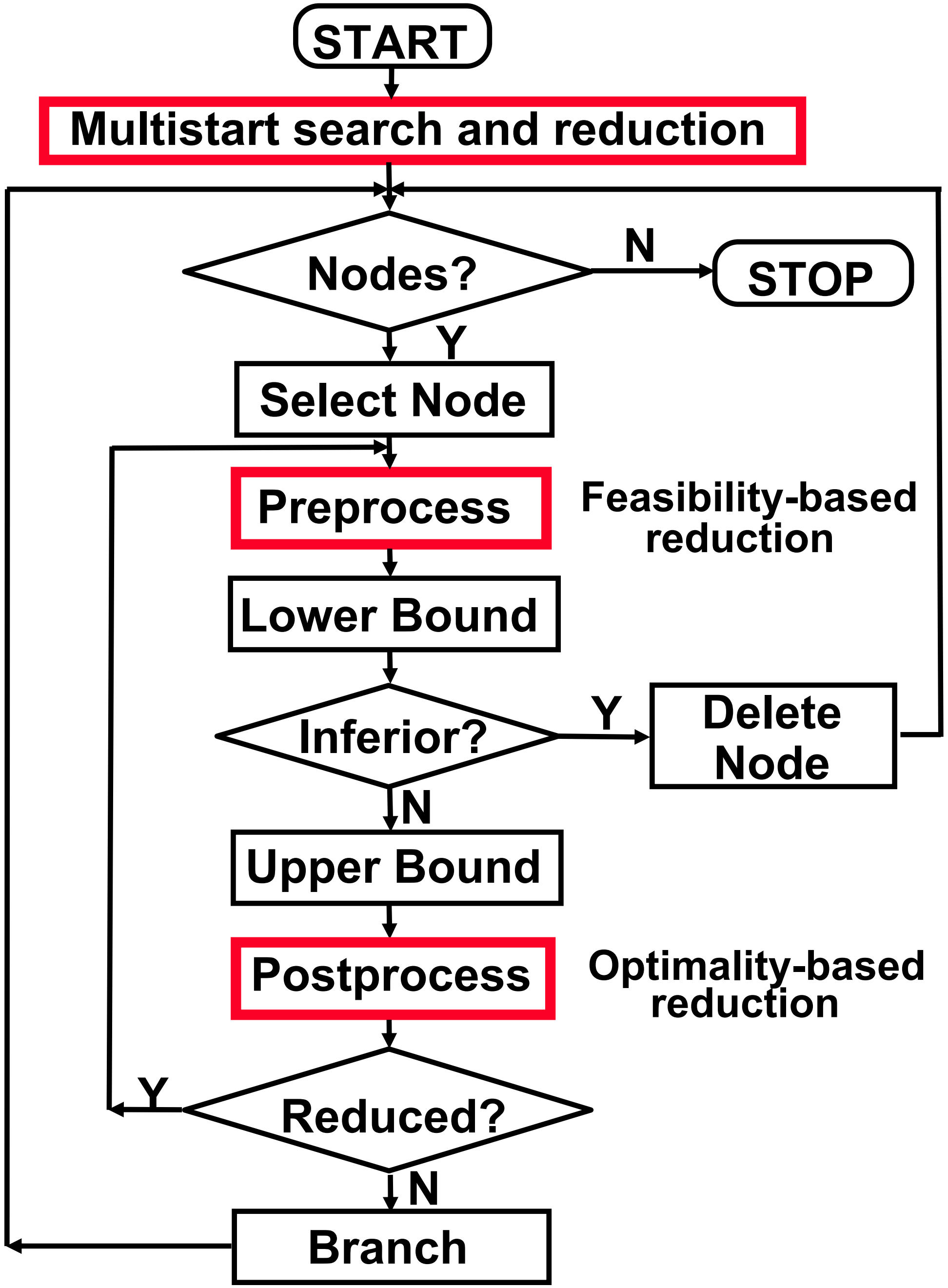
FINITE BRANCHING RULE





- **Variable selection:**
  - Typically, select variable with largest underestimating gap
  - Occasionally, select variable corresponding to largest edge
- **Point selection:**
  - Typically, at the midpoint (exhaustiveness)
  - When possible, at the best currently known solution
- **Finite isolation of global optimum**
- **Finite termination in many cases**
  - Concave minimization over polytopes
  - 2-Stage stochastic integer programming

BRANCH-AND-REDUCE



Branch-And-Reduce Optimization Navigator

Components

- Modeling language
- Preprocessor
- Data organizer
- I/O handler
- Range reduction
- Solver links
- Interval arithmetic
- Sparse matrix routines
- Automatic differentiator
- IEEE exception handler
- Debugging facilities

Capabilities

- Core module
  - Application-independent
  - Expandable
- Fully automated MINLP solver
- Application modules
  - Multiplicative programs
  - Indefinite QPs
  - Fixed-charge programs
  - Mixed-integer SDPs
  - …
- Solve relaxations using
  - CPLEX, MINOS, SNOPT, OSL, SDPA, …

- Available under GAMS and AIMMS
- Available on NEOS server

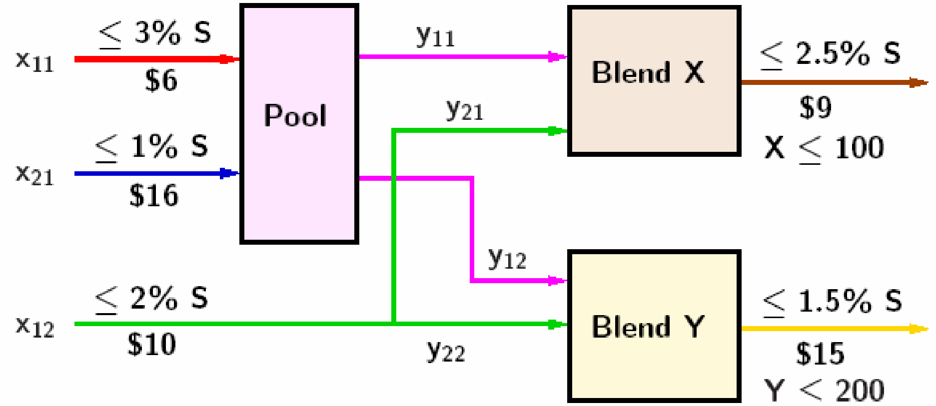
26 PROBLEMS FROM `globallib` AND `minlplib`

	Minimum	Maximum	Average
Constraints	2	513	76
Variables	4	1030	115
Discrete variables	0	432	63

EFFECT OF CUTTING PLANES

	Without cuts	With cuts	% reduction
Nodes	23,031,434	253,754	99
Nodes in memory	622,339	13,772	98
CPU hrs	76	6	93

POOLING PROBLEM: p-FORMULATION



$$\min \quad \overbrace{6x_{11} + 16x_{21} + 10x_{12}}^{\text{cost}} - \overbrace{9(y_{11} + y_{21})}^{X\text{-revenue}} - \overbrace{15(y_{12} + y_{22})}^{Y\text{-revenue}}$$

s.t. $q = \dfrac{3x_{11} + x_{21}}{y_{11} + y_{12}}$ Sulfur Mass Balance

$$x_{11} + x_{21} = y_{11} + y_{12}$$
$$x_{12} = y_{21} + y_{22}$$

Mass balance

$$\frac{qy_{11} + 2y_{21}}{y_{11} + y_{21}} \le 2.5$$
$$\frac{qy_{12} + 2y_{22}}{y_{12} + y_{22}} \le 1.5$$

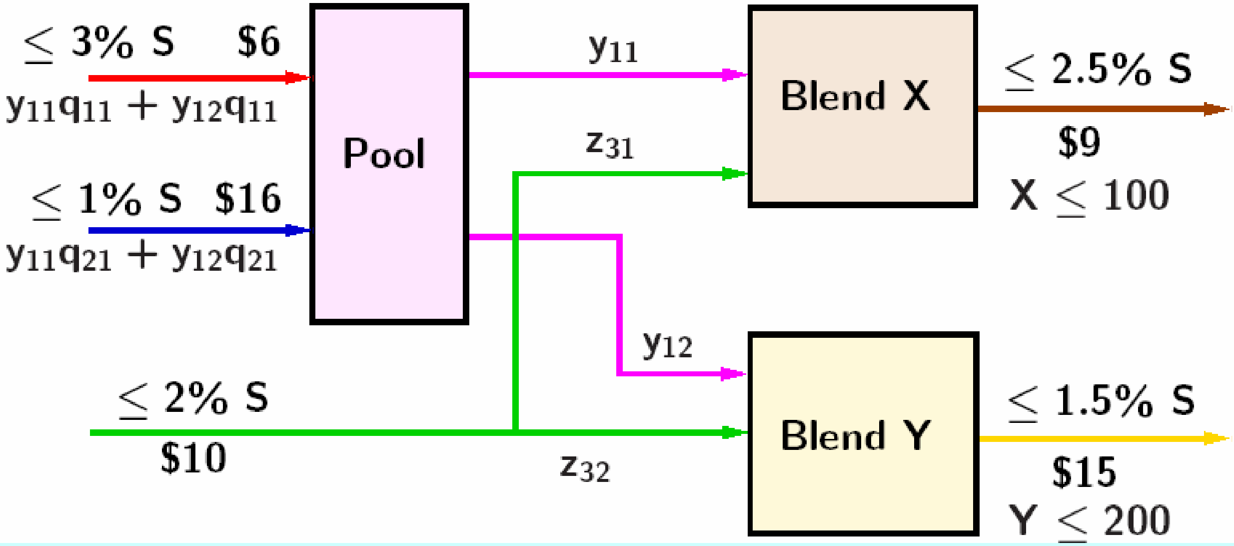
Quality Requirements

$$y_{11} + y_{21} \le 100$$
$$y_{12} + y_{22} \le 200$$

Demands

Haverly 1978

POOLING PROBLEM: q-FORMULATION



$$\min \quad \overbrace{6\,(y_{11}q_{11} + y_{12}q_{11}) + 16\,(y_{11}q_{21} + y_{12}q_{21}) + 10\,(z_{31} + z_{32})}^{\text{cost}}$$

$$- \overbrace{9(y_{11} + y_{21})}^{X\text{-revenue}} - \overbrace{15(x_{12} + x_{22})}^{Y\text{-revenue}}$$

s.t.

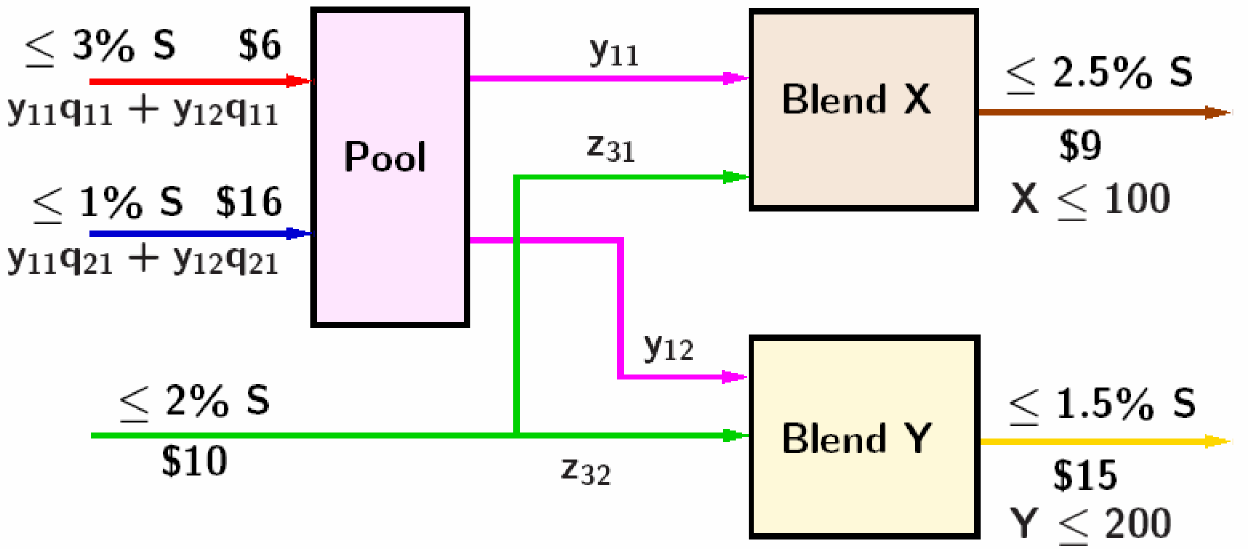
$$q_{11} + q_{21} = 1$$ Mass Balance

$$-0.5z_{31} + 3y_{11}q_{11} + y_{11}q_{21} \leq 2.5y_{11}$$
$$0.5z_{32} + 3y_{12}q_{11} + y_{12}q_{21} \leq 1.5y_{12}$$ Quality Requirements

$$y_{11} + z_{31} \leq 100$$
$$y_{12} + z_{32} \leq 200$$ Demands

Ben-Tal et al. 1994

POOLING PROBLEM: pq-FORMULATION



$$\min \quad \overbrace{6\,(y_{11}q_{11} + y_{12}q_{11}) + 16\,(y_{11}q_{21} + y_{12}q_{21}) + 10\,(z_{31} + z_{32})}^{\text{cost}}$$
$$- \overbrace{9(y_{11} + y_{21})}^{X\text{-revenue}} - \overbrace{15(x_{12} + x_{22})}^{Y\text{-revenue}}$$

s.t. $q_{11} + q_{21} = 1$ — Mass Balance

$$-0.5z_{31} + 3y_{11}q_{11} + y_{11}q_{21} \le 2.5y_{11}$$
$$0.5z_{32} + 3y_{12}q_{11} + y_{12}q_{21} \le 1.5y_{12}$$
Quality Requirements

$$y_{11} + z_{31} \le 100$$
$$y_{12} + z_{32} \le 200$$
Demands

$$y_{11}q_{11} + y_{11}q_{21} = y_{11}$$
$$y_{12}q_{11} + y_{12}q_{21} = y_{12}$$
Convexification Constraints

PRODUCT DISAGGREGATION

Consider the function:

$$\phi(x; y_1, \ldots, y_n) = a_0 + \sum_{k=1}^{n} a_k y_k + x b_0 + x \sum_{k=1}^{n} b_k y_k$$

Let

$$H = [x^L, x^U] \times \prod_{k=1}^{n} [y_k^L, y_k^U]$$

Then

$$\text{convenv}_H \, \phi = \quad a_0 + \sum_{k=1}^{n} a_k y_k + x b_0 + \sum_{k=1}^{n} \text{convenv}_{[y_k^L, y_k^U] \times [x^L, x^U]} (b_k y_k x)$$

Disaggregated formulations are tighter

LOCAL SEARCH WITH CONOPT

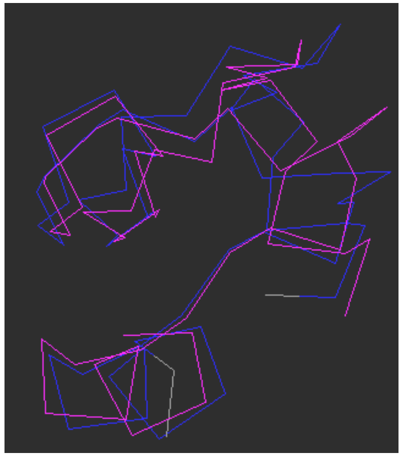
Problem	q-formulation objective	pq-formulation objective
adhya1	-68.74	-56.67
adhya2	0	0
adhya3	-65	-57.74
adhya4	-470.83	-470.83
bental4	0	0
bental5	-2900	-2700
foulds2	-1000	-600
foulds3	-6.5	-6.5
foulds4	-6	-6.5
foulds5	-7	-6.5
haverly1	-400	0
haverly2	-400	0
haverly3	-750	0
rt97	Infeasible	-4330.78

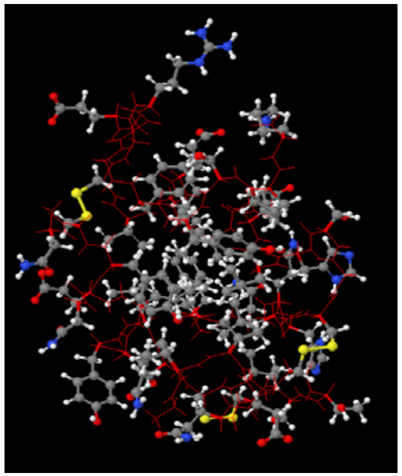
GLOBAL SEARCH WITH BARON

Problem	p-formulation		pq-formulation	
	Nodes	CPU sec	Nodes	CPU sec
adhya1	573	17	24	0.5
adhya2	501	20	17	0.5
adhya3	>9248	>1200	31	1.5
adhya4	>6129	>1200	1	1
bental4	101	0.5	1	0.5
bental5	>6445	>1200	-1	0
foulds2	1061	16	-1	0
foulds3	>348	>1200	-1	5
foulds4	>326	>1200	-1	1
foulds5	>389	>1200	-1	1
haverly1	25	0	1	0
haverly2	17	0	1	0
haverly3	3	0	1	0
rt97	5629	174	6	0.5

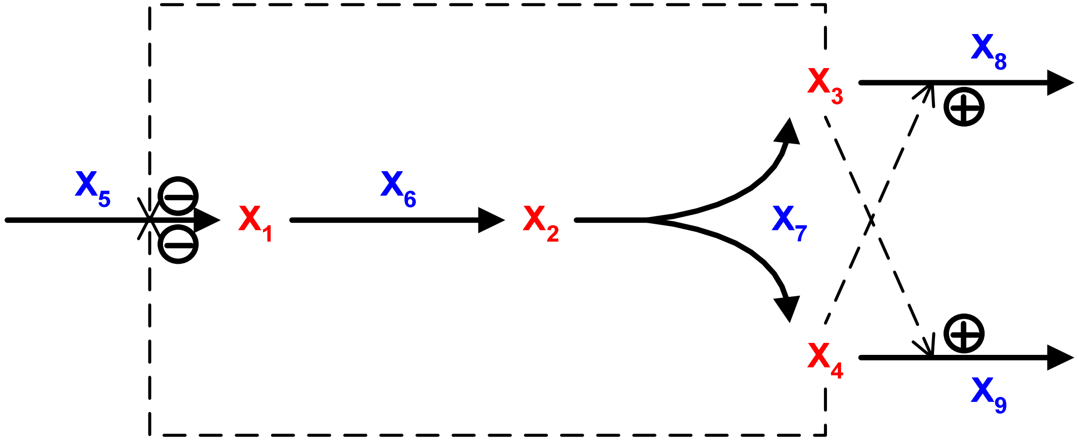
ONGOING DEVELOPMENT OF BARON

Structural Bioinformatics

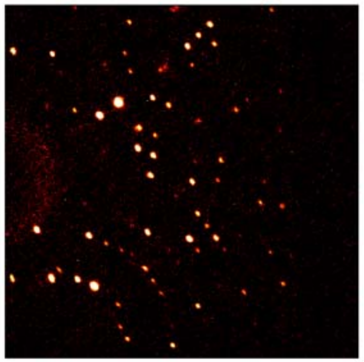




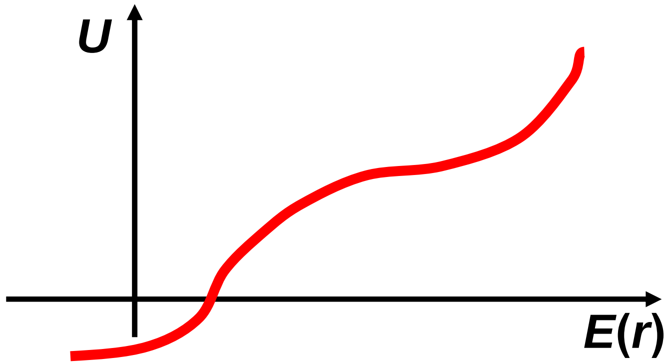
Systems biology



X-ray imaging



Portfolio optimization



BARON IN APPLICATIONS

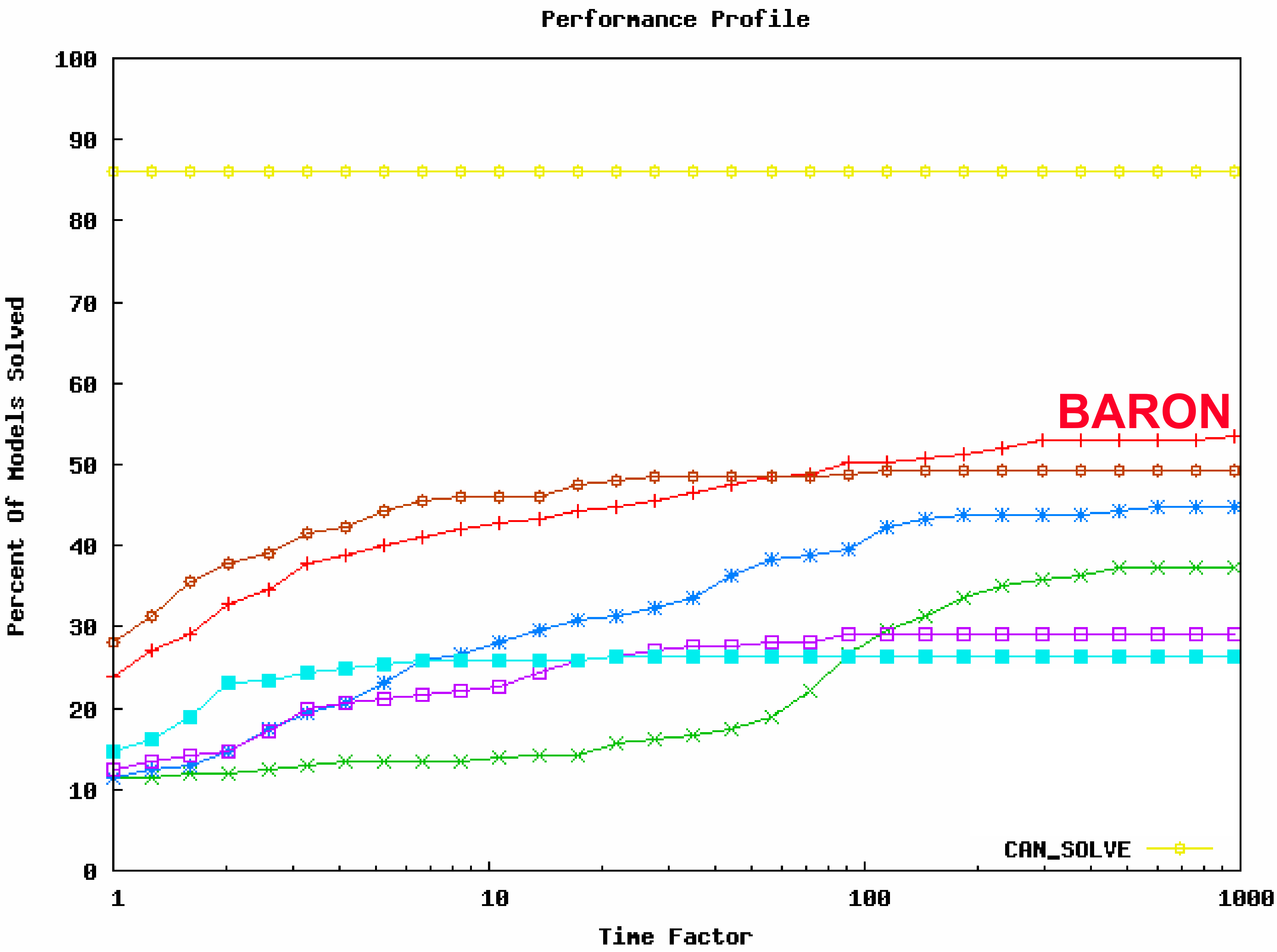
- **Development of new Runge-Kutta methods for partial differential equations**
  - **Ruuth and Spiteri, *SIAM J. Numerical Analysis,* 2004**
- **Energy policy making**
  - **Manne and Barreto, *Energy Economics,* 2004**
- **Design of metabolic pathways**
  - **Grossmann, Domach and others, *Computers & Chemical Engineering,* 2005**
- **Model estimation for automatic control**
  - **Bemporand and Ljung, *Automatica,* 2004**
- **Agricultural economics**
  - **Cabrini *et al.*, *Manufacturing and Service Operations Management,* 2005**

GLOBAL/MINLP SOFTWARE

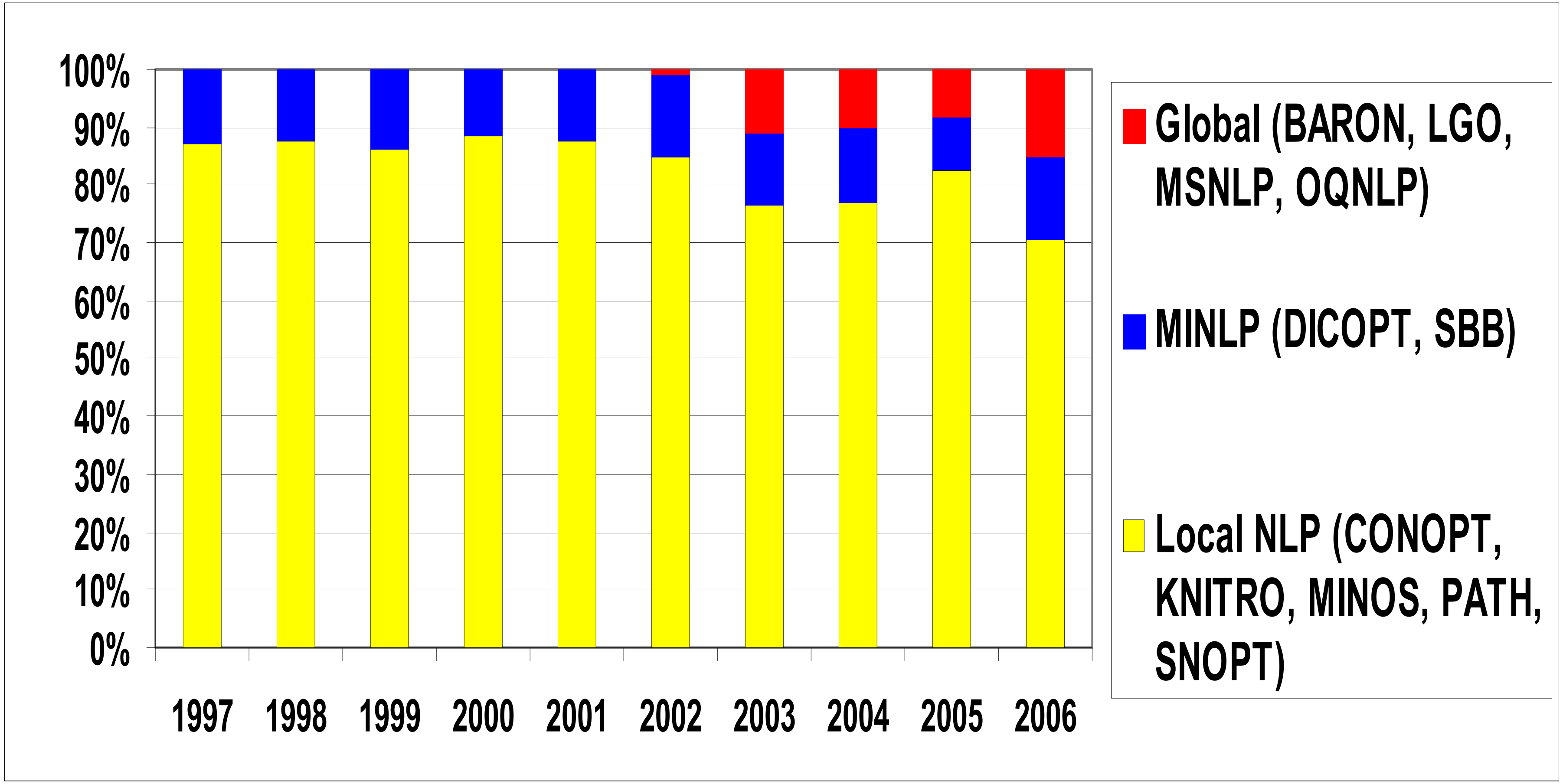
- AlphaECP—Exploits pseudoconvexity
- BARON—Branch-And-Reduce
- BONMIN—Integer programming technology (CMU/IBM)
- DICOPT—Decomposition
- GlobSol—Interval arithmetic
- Interval Solver (Frontline)—Interval solver; Excel
- LaGO—Lagrangian relaxations (COIN/OR)
- LGO—Stochastic search; black-box optimization
- LINGO—Trigonometric functions; IF-THEN-ELSE; …
- MSNLP, OQNLP—Stochastic search
- SBB—Simple branch-and-bound

NLP/MINLP NLP MINLP

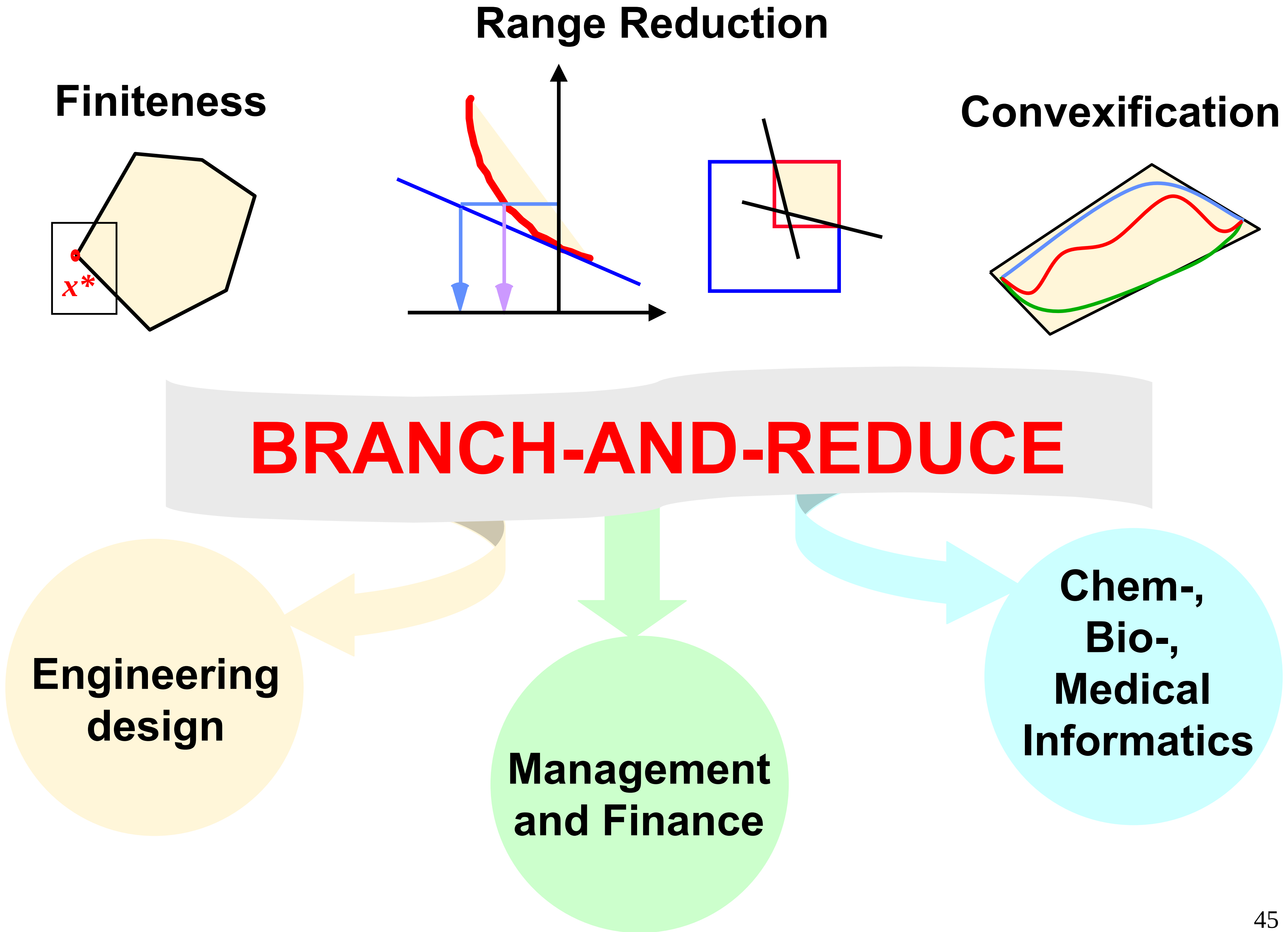
COMPARISONS ON MINLPLIB



GAMS SALES
Commercial and academic users



Data courtesy of Alex Meeraus


Range Reduction
Finiteness
Convexification
x*
BRANCH-AND-REDUCE
Engineering design
Management and Finance
Chem-, Bio-, Medical Informatics